SOLUTION FOR A NONLINEAR NUCLEAR REACTOR WITH NEGATIVE PROMPT FEEDBACK AND ONE-GROUP DELAYED NEUTRON

Nguyen van Minh

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

SOLUTION FOR A NONLINEAR NUCLEAR REACTOR
WITH NEGATIVE PROMPT FEEDBACK AND
ONE-GROUP DELAYED NEUTRON

by

Nguyen van Minh

Thesis Advisor: D. H. Nguyen

June 1973

Solution for a Nonlinear Nuclear Reactor
with negative prompt feedback and one-group delayed neutron

by

Nguyen van Minh
Lieutenant, Vietnamese Navy

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
June 1973

ABSTRACT

The nonlinear space-time neutron flux equation with negative prompt feedback and one-group delayed neutron is reduced by the use of a nonlinear transformation to a partial differential equation, in which the nonlinear term represents a small perturbation. The general procedure of solution for the resulting weakly nonlinear initial-boundary-value problem is then established by means of the method of successive approximation. Convergence of the analytical solution is also discussed. The solutions to a slab reactor core and a cylindrical reactor core are investigated here. Asymptotic stable equilibrium states are derived from each of these solutions. The present results are consistent with those obtained from previous stability analysis for the generalized buckling K greater or less than μ_o^2, the lowest eigenvalue of the associated linear Helmholtz equation.

TABLE OF CONTENTS

NOMENCLATURE

$\underline{r}$	position vector
$\underline{x}$	dimensionless position vector
t	time
v	neutron speed
D	diffusion coefficient
k	multiplication factor
k_∞	multiplication factor of an infinite reactor
k_o	multiplication factor at steady state
K	generalized buckling
C	concentration of one-group precursor
c_p	specific heat of fuel
T	temperature
T_c	temperature of coolant
T_o	temperature at steady state
e	average energy released per fission
h	convection heat transfer coefficient
C_o	concentration of one-group precursor at steady state
D	domain of a reactor core
A	surface area of a fuel element
$\forall$	volume of a fuel element
α_o	feedback coefficient
β	fraction of one-group delayed neutron
Σ_a	macroscopic absorption cross section
Σ_f	macroscopic fission cross section

τ	dimensionless time
ϕ	neutron flux distribution
ϕ_o	neutron flux at steady state
λ	decay constant of one-group precursor
ρ	fuel density
ψ	neutron flux difference, $\phi - \phi_o$
ξ	precursor concentration difference, $C - C_o$
θ	temperature difference, $T - T_o$
μ,ν	Eigenvalues

ACKNOWLEDGEMENTS

The author is extremely grateful to Dr. Dong Huu Nguyen, Professor of the Naval Postgraduate School, for his assistance, encouragement and the unselfish giving of his time.

Professor David Salinas is also due a special note of thanks for his comments in reading the rough copy of the thesis as departmental reader.

The author is deeply indebted to his wife and son for a long time that they offered encouragement and exhibited exemplary patience.

I. INTRODUCTION

The stability of nonlinear reactor kinetics has been investigated by several authors. Kastenberg and Chambré [1] have established stability theorems for the nonlinear space-time reactor kinetics equations by the method of comparison functions. Garabedian and Lynch [2] and Scalettar [3] have treated the nonlinear equations by modal expansions in space and then solved the resulting ordinary differential equations numerically. Kaplan, Margolis and Harris [4] have applied a pertubation technique as well as a modal expansion in space to obtain approximate solutions for nonseparable transients. Other numerical techniques were summarized by Kaplan, Henry, Margolis and Taylor [5].

In this thesis, the reduction of the nonlinear initial-boundary-value problem to a weakly nonlinear one is established by a nonlinear transformation. The transformed equation is now solved analytically by means of the method of successive approximations. As an application of this method, the asymptotic stable equilibrium states are discussed for an infinite slab reactor core and a finite cylindrical core.

II. DERIVATION OF THE SYSTEM EQUATIONS

A. GENERAL EQUATIONS

In this thesis, the reactor is assumed to be an one-velocity, bare, homogeneous and Newtonian cooling system with one group of delayed neutrons. The equations describing the neutron and energy distributions and the concentration for one-group delayed neutron precursor are given by Meghreblian and Holmes [6]:

$$\frac{1}{v}\frac{\partial \phi}{\partial t}(\underline{r},t) = D\nabla^2\phi(\underline{r},t) + \Sigma_a\left[(1-\beta)k-1\right]\phi(\underline{r},t) + \lambda C(\underline{r},t) \tag{2.1}$$

$$\frac{\partial C}{\partial t}(\underline{r},t) = -\lambda C(\underline{r},t) + \beta k \Sigma_a \phi(\underline{r},t) \tag{2.2}$$

$$\rho c_p \frac{\partial T}{\partial t}(\underline{r},t) = e\Sigma_f \phi(\underline{r},t) - h\frac{A}{V}\left[T(\underline{r},t) - T_c(\underline{r},t)\right] \tag{2.3}$$

where the resonance-escape and fast nonleakage probabilities to thermal are assumed to be unity. The boundary conditions of equations (2.1), (2.2) and (2.3) are given by:

$$\phi(\underline{r}_s,t) = C(\underline{r}_s,t) = T(\underline{r}_s,t) = 0 \quad \text{for } t > 0 \tag{2.4}$$

and the initial conditions are

$$\phi(\underline{r},0) = f(\underline{r})\,;\; C(\underline{r},0) = g(\underline{r})\,;\; T(\underline{r},0) = h(\underline{r}) \tag{2.5}$$

where $\underline{r}_s$ indicates all points on the boundary of the reactor.

The nonlinear effects on the kinetics of a nuclear reactor are usually described by a single parameter α_o, the feedback coefficient of the reactor. The effects of

temperature on the multiplication factor are more pronounced than the effects of temperature on the other reactor parameters. Small changes in temperature cause small changes in all the reactor parameters. However, the multiplication factor is very close to unity so that the quantity, $(1-\beta)k-1$ is very sensitive to even small changes in k. Hence, it is logical to assume that all the nonlinear coupling appears through this multiplication factor and the feedback equation is given by

$$k(T) = k_o + \alpha_o \left[T(\underline{r},t) - T_o(\underline{r}) \right] \tag{2.6}$$

The dependence of the multiplication factor on the temperature and the appearance of neutron flux ϕ in equations (2.2) and (2.3) make the system defined by equations (2.1), (2.2) and (2.3) intrinsically nonlinear and coupled.

Introducing the new dimensionless variables

$$\begin{aligned} \tau &= v\Sigma_a t \\ \underline{x} &= \sqrt{\frac{\Sigma_a}{D}}\,\underline{r} \end{aligned} \tag{2.7}$$

the above system of equations becomes:

$$\frac{\partial \phi}{\partial \tau}(\underline{x},\tau) = \nabla^2 \phi(\underline{x},\tau) + \left[(1-\beta)k-1\right]\phi(\underline{x},\tau) + \frac{\lambda}{\Sigma_a} C(\underline{x},\tau) \tag{2.8}$$

$$\frac{\partial C}{\partial \tau}(\underline{x},\tau) = -\frac{\lambda}{v\Sigma_a} C(\underline{x},\tau) + \frac{\beta k}{v}\phi(\underline{x},\tau) \tag{2.9}$$

$$\frac{\partial T}{\partial \tau}(\underline{x},\tau) = \frac{e\Sigma_f}{\rho c_p v \Sigma_a}\phi(\underline{x},\tau) - \frac{hA}{\rho c_p v \forall \Sigma_a}\left[T(\underline{x},\tau) - T_c(\underline{x})\right] \tag{2.10}$$

Defining the constants:

$$\gamma_1 = \frac{\beta k}{v} \; ; \; \gamma_2 = \frac{e\Sigma_f}{\rho c_p v \Sigma_a} \; ; \quad \gamma_3 = \frac{hA}{\rho c_p v \forall \Sigma_a} \; ; \quad \lambda_c = \frac{\lambda}{v\Sigma_a} \tag{2.11}$$

equations (2.8), (2.9) and (2.10) can be written as follows:

$$\frac{\partial \phi}{\partial \tau}(\underline{x},\tau) = \nabla^2 \phi(\underline{x},\tau) + \left[(1-\beta)k - 1\right]\phi(\underline{x},\tau) + v\lambda_c C(\underline{x},\tau) \tag{2.12}$$

$$\frac{\partial C}{\partial \tau}(\underline{x},\tau) = -\lambda_c C(\underline{x},\tau) + \gamma_1 \phi(\underline{x},\tau) \tag{2.13}$$

$$\frac{\partial T}{\partial \tau}(\underline{x},\tau) = \gamma_2 \phi(\underline{x},\tau) - \gamma_3 \left[T(\underline{x},\tau) - T_c(\underline{x})\right] \tag{2.14}$$

From these equations, putting $\frac{\partial \phi}{\partial \tau} = 0$, $\frac{\partial C}{\partial \tau} = 0$ and $\frac{\partial T}{\partial \tau} = 0$ yields the steady state equations:

$$0 = \nabla^2 \phi_0(\underline{x}) + \left[(1-\beta)k_0 - 1\right]\phi_0(\underline{x}) + v\lambda_c C_0(\underline{x}) \tag{2.15}$$

$$0 = -\lambda_c C_0(\underline{x}) + \gamma_1 \phi_0(\underline{x}) \tag{2.16}$$

$$0 = \gamma_2 \phi_0(\underline{x}) - \gamma_3 \left[T_0(\underline{x}) - T_c(\underline{x})\right] \tag{2.17}$$

which are assumed to be true for $\tau < 0$.

Subtracting equation (2.15), (2.16) and (2.17) from equations (2.12), (2.13) and (2.14) respectively yields:

$$\frac{\partial \psi}{\partial \tau}(\underline{x},\tau) = \nabla^2 \psi(\underline{x},\tau) + \left[(1-\beta)k - 1\right]\psi(\underline{x},\tau) + v\lambda_c \xi(\underline{x},\tau) \tag{2.18}$$

$$\frac{\partial \xi}{\partial \tau}(\underline{x},\tau) = -\lambda_c \xi(\underline{x},\tau) + \gamma_1 \psi(\underline{x},\tau) \tag{2.19}$$

$$\frac{\partial \theta}{\partial \tau}(\underline{x},\tau) = \gamma_2 \psi(\underline{x},\tau) - \gamma_3 \theta(\underline{x},\tau) \tag{2.20}$$

where $(1-\beta)k_0 - 1 \approx (1-\beta)k - 1$

and
$$\begin{aligned}\psi(\underline{x},\tau) &= \phi(\underline{x},\tau) - \phi_0(\underline{x}) \\ \xi(\underline{x},\tau) &= C(\underline{x},\tau) - C_0(\underline{x}) \\ \theta(\underline{x},\tau) &= T(\underline{x},\tau) - T_0(\underline{x})\end{aligned} \tag{2.21}$$

and k(T) is defined by the feedback equation (2.6)

$$k(T) = k_0 + \alpha_0 \theta(\underline{x},\tau) \tag{2.22}$$

The boundary conditions are given by:

$$\psi(\underline{x}_s,\tau) = \xi(\underline{x}_s,\tau) = \theta(\underline{x}_s,\tau) = 0 \tag{2.23}$$

where $\underline{x}_s$ designates the position vector on the boundary of the reactor and the initial conditions are:

$$\begin{aligned}\psi(\underline{x},0) &= F(\underline{x}) \\ \xi(\underline{x},0) &= 0 \\ \theta(\underline{x},0) &= 0\end{aligned} \tag{2.24}$$

Equations (2.19) and (2.20) with initial conditions (2.24) can be solved to obtain:

$$\xi(\underline{x},\tau) = \gamma_1 \int_0^\tau e^{-\lambda_c(\tau-\tau')} \psi(\underline{x},\tau')\,d\tau' \tag{2.25}$$

$$\theta(\underline{x},\tau) = \gamma_2 \int_0^\tau e^{-\gamma_3(\tau-\tau')} \psi(\underline{x},\tau')\,d\tau' \tag{2.26}$$

It is customary to rewrite equation (2.26) in another form:

$$\theta(\underline{x},\tau) = \int_0^\tau \mathcal{K}(\tau-\tau')\psi(\underline{x},\tau')d\tau' \qquad (2.27)$$

where $$\mathcal{K}(\tau-\tau') = \gamma_2 e^{-\gamma_3(\tau-\tau')} \qquad (2.28)$$

The explicit form of the kernel (2.28) depends on the feedback model under consideration [7]. Substituting equations (2.22), (2.25) and (2.27) into equation (2.18) leads to:

$$\frac{\partial\psi}{\partial\tau}(\underline{x},\tau) = \nabla^2\psi(\underline{x},\tau) + \left[(1-\beta)k_0-1\right]\psi(\underline{x},\tau) + (1-\beta)\alpha_0\psi(\underline{x},\tau)\int_0^\tau \mathcal{K}(\tau-\tau')\psi(\underline{x},\tau')d\tau'$$

$$+ \nu\lambda_c\gamma_1\int_0^\tau e^{-\lambda_c(\tau-\tau')}\psi(\underline{x},\tau')d\tau' \qquad (2.29)$$

which describes the increase in neutron flux above its steady value for one-group delayed neutron and Neutonian cooling model.

The boundary and initial conditions for equation (2.29) are given by equations (2.23) and (2.24)

$$\psi(\underline{x}_s,\tau) = 0 \qquad (2.30)$$

$$\psi(\underline{x},0) = F(\underline{x}) \qquad (2.31)$$

B. ASSUMPTIONS

Before the analysis of the reactor may be undertaken, two further assumptions must be made concerning the feedback model.

The first assumption states that the reactor temperature rises instantaneously with the flux (i.e., the prompt feedback model). The kernel $K(\tau-\tau')$ in equation (2.28) becomes [7]

$$\mathcal{K}(\tau-\tau') = \gamma_2 \delta(\tau-\tau') \tag{2.32}$$

where $\delta(\tau)$ is the Dirac delta function, hence, the temperature difference in equation (2.27) can be written as follows:

$$\theta(\underline{x},\tau) = \int_0^\tau \mathcal{K}(\tau-\tau')\psi(\underline{x},\tau')d\tau' = \gamma_2 \int_0^\tau \delta(\tau-\tau')\psi(\underline{x},\tau')d\tau'$$

or

$$\theta(\underline{x},\tau) = \gamma_2\,\psi(\underline{x},\tau) \tag{2.33}$$

The second assumption states that the temperature coefficient of a reactor is negative. This is the case of negative feedback:

$$\frac{\partial k}{\partial T} < 0 \tag{2.34}$$

Therefore, the feedback coefficient α_o in equation (2.22) is also negative:

$$\alpha_o < 0 \tag{2.35}$$

Inserting equation (2.32) into equation (2.29) provides:

$$\frac{\partial\psi}{\partial\tau}(\underline{x},\tau) = \nabla^2\psi(\underline{x},\tau) + \left[(1-\beta)k_o - 1\right]\psi(\underline{x},\tau) + (1-\beta)\alpha_o\gamma_2\psi^2(\underline{x},\tau)$$

$$+ v\lambda_c\gamma_1\int_0^\tau e^{-\lambda_c(\tau-\tau')}\psi(\underline{x},\tau')\,d\tau' \tag{2.36}$$

Let

$$(1-\beta)k_o - 1 = \kappa \tag{2.37}$$

$$(1-\beta)\alpha_o\gamma_2 = -\alpha \quad (\alpha > 0) \tag{2.38}$$

$$v\lambda_c\gamma_1 = \beta^* \tag{2.39}$$

Equation (2.36) yields:

$$\boxed{\frac{\partial\psi}{\partial\tau}(\underline{x},\tau) = \nabla^2\psi(\underline{x},\tau) + K\psi(\underline{x},\tau) - \alpha\psi^2(\underline{x},\tau) + \beta^*\int_0^\tau e^{-\lambda_c(\tau-\tau')}\psi(\underline{x},\tau')d\tau'} \tag{2.40}$$

and its initial-boundary conditions are given by equation (2.30) and (2.31). Equation (2.40) describes the increase of neutron flux above its steady value for a negative prompt feedback and one-group delayed neutron model, the solution of which will be established in the next section.

III. REDUCTION OF A NONLINEAR SYSTEM TO A LINEAR SYSTEM

A. NONLINEAR TRANSFORMATION

The Wilhelm's transformation [8] is considered:

$$\psi(\underline{x},\tau) = \frac{e^{\kappa\tau}u(\underline{x},\tau)}{1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u(\underline{x},\tau)} \tag{3.1}$$

or

$$u(\underline{x},\tau) = \frac{\psi(\underline{x},\tau)}{e^{\kappa\tau}-\frac{\alpha}{\kappa}(e^{\kappa\tau}-1)\psi(\underline{x},\tau)} \tag{3.2}$$

Since the neutron flux difference $\psi(\underline{x},\tau)$ is non-negative, it is logical to assume that:

$$u(\underline{x},\tau) \geq 0 \quad \text{for } \underline{x} \text{ in } \mathcal{D} \text{ and } \tau > 0$$

hence,

$$1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u(\underline{x},\tau) > 0 \tag{3.3}$$

The nonlinear transformation (3.1) is generally valid, since it provides a unique interrelation between $\psi(\underline{x},\tau) \geq 0$ and $u(\underline{x},\tau) \geq 0$, which is free from singularities for any point $(\underline{x},\tau)$. Differentiating equation (3.1) yields:

$$\frac{\partial\psi}{\partial\tau} = \left(\frac{\partial u}{\partial\tau} + \kappa u - \alpha u^2\right)\frac{e^{\kappa\tau}}{\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2} \tag{3.4}$$

$$\nabla^2\psi = \frac{e^{\kappa\tau}\nabla^2 u}{\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2} + \frac{2\alpha(1-e^{\kappa\tau})e^{\kappa\tau}(\nabla u)^2}{\kappa\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^3} \tag{3.5}$$

Substituting equations (3.1), (3.4) and (3.5) into equation (2.40) provides:

$$\left(\frac{\partial u}{\partial \tau} + \kappa u - \alpha u^2\right)\frac{e^{\kappa\tau}}{\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2} = \frac{e^{\kappa\tau}\nabla^2 u}{\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2} + \frac{2\alpha(1-e^{\kappa\tau})e^{\kappa\tau}(\nabla u)^2}{\kappa\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^3}$$

$$+ \frac{\kappa e^{\kappa\tau}u}{1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u} - \frac{\alpha e^{2\kappa\tau}u^2}{\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2}$$

$$+ \beta^*\int_0^\tau e^{-\lambda_c(\tau-\tau')}\frac{e^{\kappa\tau'}u(\underline{x},\tau')\,d\tau'}{1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau'})u(\underline{x},\tau')} \tag{3.6}$$

Dividing both sides by $\dfrac{e^{\kappa\tau}}{[1 - \frac{\alpha}{\kappa}(1-e^{\kappa\tau})u]^2}$ and simplifying give:

$$\frac{\partial u}{\partial \tau} = \nabla^2 u - \frac{2\alpha(e^{\kappa\tau}-1)(\nabla u)^2}{\kappa\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]} + \beta^*\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2\int_0^\tau e^{-\kappa_0(\tau-\tau')}\frac{u(\underline{x},\tau')\,d\tau'}{1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau'})u(\underline{x},\tau')} \tag{3.7}$$

where $$\kappa_0 = \kappa + \lambda_c > 0 \tag{3.8}$$

$$\kappa > 0 \quad \text{and} \quad \lambda_c > 0$$

Let $$\eta_1 = -\frac{2\alpha(e^{\kappa\tau}-1)(\nabla u)^2}{\kappa\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]} \tag{3.9a}$$

and $$\eta_2 = \beta^*\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u\right]^2\int_0^\tau e^{-\kappa_0(\tau-\tau')}\frac{u(\underline{x},\tau')\,d\tau'}{1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau'})u(\underline{x},\tau')} \tag{3.9b}$$

Equation (3.7) becomes:

$$\frac{\partial u}{\partial \tau}(\underline{x},\tau) = \nabla^2 u(\underline{x},\tau) + \eta_1(\underline{x},\tau) + \eta_2(\underline{x},\tau) \tag{3.10}$$

The initial and boundary conditions are:

$$u(\underline{x}_s, \tau) = 0 \tag{3.11}$$

$$u(\underline{x}, 0) = \psi(\underline{x}, 0) = F(\underline{x}) \tag{3.12}$$

B. COMPARISON OF TERMS OF THE TRANSFORMED EQUATION

Before solving equation (3.10) by successive approximations method, it is necessary to impose some conditions on the terms η_1 and η_2 such that they are much smaller than the other terms in equation (3.10). A comparison of the terms of equation (3.6) indicates that the terms $\left(\frac{\partial u}{\partial \tau}\right)e^{K\tau}$ and $(\nabla^2 u)e^{K\tau}$ have the same order of magnitude as the term $Kue^{K\tau}$. Therefore, the nonlinear terms η_1 and η_2 are negligibly small in equation (3.10) for any time $0 \leq \tau \leq \infty$ if and only if the ratios $\left|\frac{\eta_1}{Kue^{K\tau}}\right|$ and $\left|\frac{\eta_2}{Kue^{K\tau}}\right|$ are small compared to unity, i.e., for $\underline{x}$ in D and $0 \leq \tau \leq \infty$:

$$\left|\frac{\eta_1}{Kue^{K\tau}}\right| = \frac{2\alpha(e^{K\tau}-1)(\nabla u)^2}{K^2 e^{K\tau}\left[1-\frac{\alpha}{K}(1-e^{K\tau})u\right]u} \ll 1 \tag{3.13}$$

and

$$\left|\frac{\eta_2}{Kue^{K\tau}}\right| = \frac{\beta^*}{Kue^{K\tau}}\left[1-\frac{\alpha}{K}(1-e^{K\tau})u\right]^2 \int_0^{\tau} e^{-K_0(\tau-\tau')} \frac{u(\underline{x},\tau')d\tau'}{1-\frac{\alpha}{K}(1-e^{K\tau'})u(\underline{x},\tau')} \ll 1 \tag{3.14}$$

Since u and ∇u are regular, as τ tends to zero, the numerators of expressions (3.13) and (3.14) approach to zero, hence ,

$$\lim_{\tau \to 0} \left| \frac{\eta_1}{K u e^{K\tau}} \right| \longrightarrow 0$$

$$\lim_{\tau \to 0} \left| \frac{\eta_2}{K u e^{K\tau}} \right| \longrightarrow 0$$

For the time being, it is assumed that the conditions (3.13) and (3.14) are satisfied. They will be verified after the solution to equation (3.10) is obtained in the next section.

C. LINEAR EQUATION TO BE SOLVED

Let $$\mathcal{E} = \eta_1 + \eta_2 \tag{3.15}$$

Equation (3.10) becomes:

$$\frac{\partial u(\underline{x},\tau)}{\partial \tau} = \nabla^2 u(\underline{x},\tau) + \mathcal{E}[\tau, u(\underline{x},\tau)] \tag{3.16}$$

$$u(\underline{x}_s,\tau) = 0 \tag{3.17}$$

$$u(\underline{x},0) = F(\underline{x}) \tag{3.18}$$

The transformed equation (3.16) now can be solved by the method of successive approximations. In this approach, the small nonlinear term $\mathcal{E}[\tau,u(x,\tau)]$ is treated as a perturbation. Hence, the initial-boundary-value problem is written as follows:

$$\frac{\partial u_i}{\partial \tau} = \nabla^2 u_i + \mathcal{E}_i[\tau, u_{i-1}(\underline{x},\tau)] \tag{3.19}$$

$$u_i(\underline{x}_s,\tau) = 0 \tag{3.20}$$

$$u_i(\underline{x},0) = F(\underline{x}) \tag{3.21}$$

where $$\mathcal{E}_0 = 0$$

and

$$\mathcal{E}_i = \eta_1 + \eta_2 = -\frac{2\alpha(e^{\kappa\tau}-1)(\nabla u_{i-1})^2}{\kappa\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u_{i-1}\right]} + \beta^*\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u_{i-1}\right]^2 \int_0^{\tau} \frac{e^{-\kappa_0(\tau-\tau')}u_{i-1}(\underline{x},\tau')\,d\tau'}{1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau'})u_{i-1}(\underline{x},\tau')} \quad (3.22)$$

thus, the reduction of the nonlinear initial-boundary-value problem, equation (2.40) to a linear one, equation (3.19) is established by the nonlinear transformation, provided that the conditions (3.13) and (3.14) are satisfied.

IV. ANALYSIS OF A SLAB REACTOR CORE

A. SOLUTION

Consider an infinite homogeneous slab [Fig. 1]. The medium is of infinite extent in both y and z directions and of width 2a in the x direction (a is the dimensionless extrapolated half-thickness of the slab); the origin of the x axis is placed at the center of the slab. These specifications reduce the spatial dependence of the flux to variations in x alone.

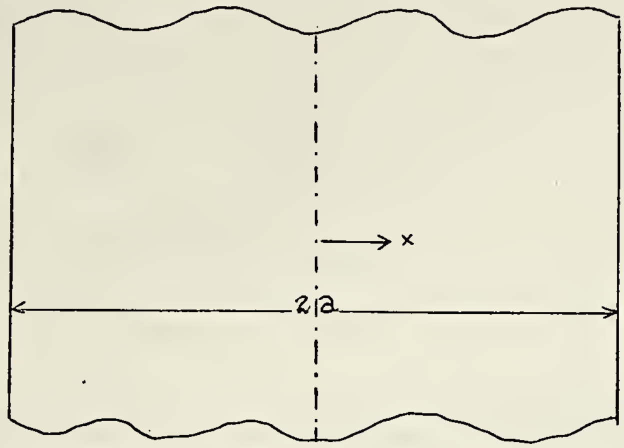

Figure 1. A Bare Slab Reactor.

Therefore, equations (3.19), (3.20) and (3.21) become:

$$\frac{\partial u_i}{\partial \tau}(x,\tau) = \frac{\partial^2 u_i}{\partial x^2}(x,\tau) + \mathcal{E}_i$$

$$u_i(\pm a,\tau) = 0 \tag{4.1}$$

$$u_i(x,0) = F(x)$$

where $$\mathcal{E}_o = 0$$

and

$$\mathcal{E}_i = -\frac{2\alpha(e^{K\tau}-1)\left(\frac{\partial u_{i-1}}{\partial x}\right)^2}{K\left[1-\frac{\alpha}{K}(1-e^{K\tau})u_{i-1}\right]} + \beta^*\left[1-\frac{\alpha}{K}(1-e^{K\tau})u_{i-1}\right]^2\int_0^\tau e^{-K_0(\tau-\tau')}\frac{u_{i-1}(x,\tau')d\tau'}{1-\frac{\alpha}{K}(1-e^{K\tau'})u_{i-1}(x,\tau')} \quad (i=1,2,3,\ldots) \tag{4.2}$$

As seen in Section III, equation (4.1) will be solved by the method of successive approximations where the zeroth approximation corresponds to $\mathcal{E}_o = 0$.

1. <u>Zeroth Approximation</u> ($\mathcal{E}_o = 0$)

   The zeroth approximation $u_o(x,\tau)$ is defined by the following equations:

$$\begin{aligned} \frac{\partial u_o}{\partial \tau} &= \frac{\partial^2 u_o}{\partial x^2} \\ u_o(\pm a,\tau) &= 0 \\ u_o(x,0) &= F(x) \end{aligned} \tag{4.3}$$

Equation (4.3) is of the homogeneous, parabolic type that can be solved by either separation of variables or using the Green's function [9]. Throughout this work, the homogeneous equations will be solved by separation of variables.

Let $$u_o(x,\tau) = X(x)\,T(\tau) \tag{4.4}$$

hence, $$\frac{\partial u_o}{\partial \tau} = X\,T' \quad \text{and} \quad \frac{\partial^2 u_o}{\partial x^2} = X''\,T \tag{4.5}$$

Substituting equations (4.5) into equation (4.3) and then dividing by XT yield:

$$\frac{T'}{T} = \frac{X''}{X} = -\mu^2 \tag{4.6}$$

where μ is the eigenvalue to be determined from the homogeneous boundary conditions:

$$X(\pm a) = 0 \tag{4.7}$$

Equations (4.6) can be written as 2 ordinary differential equations:

$$X'' + \mu^2 X = 0 \tag{4.8}$$

$$T' + \mu^2 T = 0 \tag{4.9}$$

The solution to equation (4.8) is

$$X(x) = C_1 \cos\mu x + C_2 \sin\mu x \tag{4.10}$$

where c_1 and c_2 are 2 arbitrary constants. The boundary conditions (4.7) give

$$C_1 \cos\mu a + C_2 \sin\mu a = 0 \tag{4.11}$$

and $$C_1 \cos\mu a - C_2 \sin\mu a = 0 \tag{4.12}$$

Subtracting equation (4.12) from (4.11) leads to

$$2C_2 \sin\mu a = 0$$

This implies $c_2 = 0$ or $\sin\mu a = 0$.

If $\sin\mu a = 0$, then $\mu a = n\pi$, $n = 0,1,2,...$ and equation (4.11) provides

$$C_1 \cos n\pi = 0$$

or $$c_1 = 0$$

and the general solution (4.10) is of the form:

$$X(x) = c_2 \sin \frac{n\pi}{a} x \tag{4.13}$$

This solution is unacceptable since $u_o(x,\tau)$ is non-negative for $-a \leq x \leq a$, hence,

$$c_2 = 0 \tag{4.14}$$

and equation (4.11) gives

$$c_1 \cos \mu a = 0$$

Since $c_1 \neq 0$, then

$$\cos \mu a = 0$$

or the eigenvalues are defined by

$$\mu_n = (2n+1)\frac{\pi}{2a} \quad , n = 0, 1, 2, \cdots \tag{4.15}$$

When equations (4.14) and (4.15) are inserted into equation (4.10) the solution of equation (4.8) is obtained as follows:

$$X(x) = c_1 \cos (2n+1)\frac{\pi}{2a} x \tag{4.16}$$

The solution to equation (4.9) is

$$T(\tau) = c_3 e^{-(2n+1)^2 \frac{\pi^2}{4a^2} \tau} \tag{4.17}$$

where c_3 is an arbitrary constant. Combining equations (4.16) and (4.17) gives the general solution to equation (4.3)

$$u_o(x,\tau) = \sum_{n=0}^{\infty} B_n e^{-\mu_n^2 \tau} \cos\mu_n x \tag{4.18}$$

where $_n$ are the eigenvalues given by expression (4.15). The coefficients B_n are determined from the initial condition: $u_o(x,o) = F(x)$

$$u_o(x,o) = F(x) = \sum_{n=0}^{\infty} B_n \cos\mu_n x \tag{4.19}$$

The above equation indicates that the B_n's should be the coefficients of the Fourier's series expansion of function $F(x)$, i.e.,

$$B_n = \frac{1}{a}\int_{-a}^{a} F(\xi)\cos\mu_n\xi\, d\xi \tag{4.20}$$

Replacing B_n in equation (4.18) by its value in equation (4.20) yields

$$u_o(x,\tau) = \sum_{n=0}^{\infty}\left[\frac{1}{a} e^{-\mu_n^2\tau}\cos\mu_n x \int_{-a}^{a} F(\xi)\cos\mu_n\xi\, d\xi\right]$$

Since the series $\sum_{n=0}^{\infty} e^{-\mu_n^2\tau} \cos\mu_n x \cos\mu_n\xi$ is uniformly convergent for $-a \le x \le a$ and $0 < \tau \le \infty$ [cf. appendix A] the interchange of signs Σ and $\int$ is possible, then,

$$u_o(x,\tau) = \int_{-a}^{a}\left[F(\xi)\frac{1}{a}\sum_{n=0}^{\infty} e^{-\mu_n^2\tau}\cos\mu_n x \cos\mu_n\xi\right] d\xi \tag{4.21}$$

Let

$$G(x,\xi,\tau) = \frac{1}{a}\sum_{n=0}^{\infty} e^{-\mu_n^2\tau}\cos\mu_n x \cos\mu_n\xi \tag{4.22}$$

Equation (4.21) can be written in the form:

$$\boxed{u_o(x,\tau) = \int_{-a}^{a} G(x,\xi,\tau)\,F(\xi)\,d\xi} \tag{4.23}$$

where $G(x,\xi,\tau)$ is usually called the Green's function [8] associated with equation (4.3)

2. First approximation

The first approximation is the solution to the following equation:

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial^2 u_1}{\partial x^2} + \mathcal{E}_1(x,\tau)$$

$$u_1(\pm a,\tau) = 0 \tag{4.24}$$

$$u_1(x,0) = F(x)$$

where $\mathcal{E}_1$ is obtained from equation (4.2) by making $i = 1$ and substituting equation (4.23) for $u_o(x,\tau)$. The solution of equation (4.24) is the superposition of two solutions, u_o and v to the following equations:

$$\frac{\partial u_o}{\partial \tau} = \frac{\partial^2 u_o}{\partial x^2} \tag{4.25}$$

$$u_o(\pm a,\tau) = 0$$

$$u_o(x,0) = F(x)$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \mathcal{E}_1(x,\tau) \tag{4.26}$$

$$v(\pm a,\tau) = 0$$

$$v(x,0) = 0$$

$$u_1 = u_o + v \tag{4.27}$$

Equation (4.25) is identical to equation (4.3), hence the solution $u_o(x,\tau)$ is given by expressions (4.18) or (4.23).

The solution to equation (4.26) is now assumed to be:

$$v(x,\tau) = \sum_{n=0}^{\infty} A_n(\tau)\cos\mu_n x \qquad (4.28)$$

the μ_n's are determined by equation (4.15). The source term $\mathcal{E}_1(x,\tau)$ in equation (4.26) is then expanded in terms of the eigenfunctions, $\cos\mu_n x$:

$$\mathcal{E}_1(x,\tau) = \sum_{n=0}^{\infty} \mathcal{E}_n'(\tau)\cos\mu_n x \qquad (4.29)$$

where
$$\mathcal{E}_n'(\tau) = \frac{1}{a}\int_{-a}^{a}\mathcal{E}_1(\xi,\tau)\cos\mu_n\xi\, d\xi \qquad (4.30)$$

Substitute equations (4.28) and (4.29) into equation (4.26) to obtain:

$$\sum_{n=0}^{\infty}\dot{A}_n(\tau)\cos\mu_n x = -\sum_{n=0}^{\infty}\mu_n^2 A_n(\tau)\cos\mu_n x + \sum_{n=0}^{\infty}\mathcal{E}_n'(\tau)\cos\mu_n x$$

where $\dot{A}_n(\tau) = \dfrac{dA_n}{d\tau}$

or
$$\sum_{n=0}^{\infty}\left[\dot{A}_n(\tau)+\mu_n^2 A_n(\tau)-\mathcal{E}_n'(\tau)\right]\cos\mu_n x = 0 \qquad (4.31)$$

Since the eigenfunctions, $\cos\mu_n x$ are linearly independent, equality (4.31) implies that:

$$\dot{A}_n(\tau)+\mu_n^2 A_n(\tau) = \mathcal{E}_n'(\tau) \qquad (4.32)$$

$$n = 0, 1, 2, \ldots$$

The initial condition is given by:

$$V(x,0) = 0 = \sum_{n=0}^{\infty} A_n(0)\cos\mu_n x$$

which is equivalent to:

$$A_n(0) = 0 \quad , \quad n = 0,1,2,\dots \tag{4.33}$$

Multiplying equation (4.32) by $e^{\mu_n^2\tau}$ and integrating from 0 to τ yields the solution to this equation subject to the initial condition (4.33):

$$A_n(\tau) = e^{-\mu_n^2\tau}\int_0^{\tau} \xi_n^{1}(t)e^{\mu_n^2 t}dt \tag{4.34}$$

where $\xi_n^1(t)$ is defined by expression (4.30)

$$A_n(\tau) = e^{-\mu_n^2\tau}\int_0^{\tau} e^{\mu_n^2 t}\left[\frac{1}{a}\int_{-a}^{a}\xi_1(\xi,t)\cos\mu_n\xi\, d\xi\right]dt$$

or

$$A_n(\tau) = \int_0^{\tau}\int_{-a}^{a}\frac{1}{a}e^{-\mu_n^2(\tau-t)}\xi_1(\xi,t)\cos\mu_n\xi\, d\xi\, dt \tag{4.35}$$

$$n = 0,1,2,3,\dots$$

Inserting equation (4.35) into expression (4.28) leads to the solution of equation (4.26)

$$V(x,\tau) = \sum_{n=0}^{\infty}\left[\cos\mu_n x\int_0^{\tau}\int_{-a}^{a}\frac{1}{a}e^{-\mu_n^2(\tau-t)}\xi_1(\xi,t)\cos\mu_n\xi\, d\xi\, dt\right]$$

Since $\xi_1(\xi,t)$ is well-behaved for regular $u(\xi,t)$, the uniform convergence of the series $\sum_{n=0}^{\infty} e^{-\mu_n^2(\tau-t)} \cos\mu_n x \cos\mu_n\xi$, [cf. appendix A] allows the interchange of the symbols of summation and integration in the above equation:

$$V(x,\tau) = \int_0^\tau \int_{-a}^{a} \left[\mathcal{E}_1(\xi,t) \frac{1}{a} \sum_{n=0}^{\infty} e^{-\mu_n^2(\tau-t)} \cos\mu_n x \cos\mu_n \xi \right] d\xi\, dt \tag{4.36}$$

From the definition of the Green's function, equation (4.22), equation (4.36) can be written as

$$V(x,\tau) = \int_0^\tau \int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\, d\xi\, dt \tag{4.37}$$

As seen from equation (4.27), the first approximation is given by

$$\boxed{u_1(x,\tau) = u_0(x,\tau) + \int_0^\tau \int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t) d\xi\, dt} \tag{4.38}$$

3. i^{th} Approximation

It is noted that equation (4.1) for the i^{th} approximation is similar to equation (4.24) for the first approximation with the same initial boundary conditions. Furthermore, the Green's function associated with a differential equation only depends on the form of that equation and the boundary conditions [10]. Therefore, the solutions to all possible equations (4.1) with different $\mathcal{E}_i(x,\tau)$ can be deduced from expression (4.38) by replacing $\mathcal{E}_i(x,\tau)$ for $\mathcal{E}_1(x,\tau)$

$$\boxed{\begin{aligned} u_i(x,\tau) &= u_0(x,\tau) + \int_0^\tau \int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_i(\xi,t) d\xi\, dt \\ i &= 0, 1, 2, \dots \end{aligned}} \tag{4.39}$$

where $\mathcal{E}_i(\xi,t)$ is given by equations (4.2). Once $u_i(x,\tau)$ is evaluated from equation (4.39), the i^{th} approximation of

the neutron flux difference will be calculated from the non-linear transformation (3.1):

$$\Psi_i(x,\tau) = \frac{e^{\kappa\tau} u_i(x,\tau)}{1 - \frac{\alpha}{\kappa}(1-e^{\kappa\tau}) u_i(x,\tau)} \tag{4.40}$$

Thus, the analytical solution for an infinite slab reactor core with negative prompt feedback and one-group delayed neutron is found.

At this point, it is necessary to verify the conditions (3.13) and (3.14) and the convergence of the sequence $\{u_i(x,\tau)\}$. All these questions are discussed in the next subsections.

B. VERIFICATION OF ASSUMPTIONS

As shown in part III-B, the conditions (3.13) and (3.14) are justified at $\tau = 0$ and in its neighborhood provided that the function $u(x,\tau)$ and its derivatives $\frac{\partial u}{\partial x}$ are continuous. These conditions will be verified for any τ.

For the sake of simplicity, the zeroth and the first approximations are used here to calculate the values of $\mathcal{E}(x,t)$ and $u(x,\tau)$ respectively.

Expression (3.22) yields the value of $\mathcal{E}_1(x,\tau)$:

$$\mathcal{E}_1 = \eta_1' + \eta_2'$$

where

$$\eta_1' = -\frac{2\alpha(e^{\kappa\tau}-1)\left(\frac{\partial u_0}{\partial x}\right)^2}{\kappa\left[1-\frac{\alpha}{\kappa}(1-e^{\kappa\tau})u_0\right]} \tag{4.41}$$

and

$$\eta_2' = \beta^*\left[1 - \frac{\alpha}{K}(1 - e^{K\tau})u_o\right]^2 \int_0^\tau \frac{e^{-K_o(\tau-\tau')} u_o(x,\tau')\,d\tau'}{1 - \frac{\alpha}{K}(1 - e^{K\tau'})u_o(x,\tau')} \qquad (4.42)$$

The fundamental mode of $u_o(x,\tau)$ is given by equation (4.18)

$$u_o(x,\tau) = B_o e^{-\mu_o^2\tau}\cos\mu_o x$$

and so

$$\frac{\partial u_o}{\partial x}(x,\tau) = -B_o\mu_o e^{-\mu_o^2\tau}\sin\mu_o x$$

Let $$a_1(x) = \frac{\alpha}{K} B_o \cos\mu_o x \qquad (4.43)$$

The function $a_1(x)$ is positively small for $-a < x < a$ and of order of $\frac{\alpha}{K} B_o$ since α is very small (or order about 10^{-21})

$$0 < a_1 \approx \frac{\alpha}{K} B_o \ll 1 \qquad (4.44)$$

Expression (4.41) and (4.42) are written as:

$$|\eta_1'| = \frac{2\alpha B_o^2 \mu_o^2 \left[e^{(K-2\mu_o^2)\tau} - e^{-2\mu_o^2\tau}\right]\sin^2\mu_o x}{K\left[1 - a_1(x)e^{-\mu_o^2\tau} + a_1(x)e^{(K-\mu_o^2)\tau}\right]} \qquad (4.45)$$

$$\eta_2' = \beta^* B_o e^{-K_o\tau}\left[1 - a_1 e^{-\mu_o^2\tau} + a_1 e^{(K-\mu_o^2)\tau}\right]^2 \cos\mu_o x \int_0^\tau \frac{e^{(K_o-\mu_o^2)\tau'}\,d\tau'}{1 - a_1 e^{-\mu_o^2\tau'} + a_1 e^{(K-\mu_o^2)\tau'}} \qquad (4.46)$$

If the term $a_1 e^{-\mu_o^2\tau}$ in equations (4.45) and (4.46) is neglected for simplicity, it follows that:

$$|\eta_1'| \leq \frac{2\alpha B_o^2 \mu_o^2 e^{(K-2\mu_o^2)\tau}}{K\left[1 + a_1(x) e^{(K-\mu_o^2)\tau}\right]} \tag{4.47}$$

$$|\eta_2'| \leq \beta^* B_o e^{-K_o\tau}\left[1 + a_1(x) e^{(K-\mu_o^2)\tau}\right]^2 \int_0^\tau \frac{e^{(K_o-\mu_o^2)\tau'}}{1 + a_1(x) e^{(K-\mu_o^2)\tau'}} d\tau' \tag{4.48}$$

a. Case $K = \mu_o^2$

Inequality (4.47) becomes:

$$|\eta_1'| \leq \frac{2\alpha B_o^2 \mu_o^2 e^{-\mu_o^2\tau}}{K[1 + a_1]} < \frac{2\alpha}{K} B_o^2 \mu_o^2 e^{-\mu_o^2\tau}$$

or

$$\left|\frac{\eta_1'}{K u e^{K\tau}}\right| < \frac{2\alpha B_o}{K} e^{-K\tau} < \frac{2\alpha B_o}{K} \quad \text{for any} \quad \tau \geq 0$$

Assume $\alpha \simeq 10^{-21}$, $B_o \simeq 10^{10}$, $K = 0.09$, $\lambda_c = 10^{-6}$, then

$$\left|\frac{\eta_1'}{K u e^{K\tau}}\right| < \frac{2 \times 10^{-21} \times 10^{10}}{0.09} = \frac{2}{9} \times 10^{-9}$$

or

$$\left|\frac{\eta_1'}{K u e^{K\tau}}\right| \ll 1 \quad \text{for any} \quad \tau \geq 0$$

In the same manner, expression (4.48) is equivalent to

$$|\eta_2'| \leq \beta^* B_o e^{-K_o\tau}(1 + a_1)^2 \int_0^\tau \frac{e^{(K_o-\mu_o^2)\tau'}}{(1 + a_1)} d\tau' \approx \beta^* B_o e^{-K_o\tau} \int_0^\tau e^{\lambda_c\tau'} d\tau'$$

$$\leq \frac{\beta^* B_o}{\lambda_c} e^{-K_o\tau}(e^{\lambda_c\tau} - 1) < \frac{\beta^* B_o}{\lambda_c} e^{-K_o\tau} e^{\lambda_c\tau}$$

Replacing K_o by $(K+\ _c)$ in the above inequality yields:

$$|\eta_2'| < \frac{\beta^* \beta_0}{\lambda_c} e^{-K\tau}$$

and

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{\beta^*}{\lambda_c K} e^{(-K+\mu_0^2)\tau} e^{-K\tau} = \frac{\beta^*}{\lambda_c K} e^{-K\tau}$$

or

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{\beta^*}{\lambda_c K} \quad \text{for any} \quad \tau \geq 0$$

Let $\beta^* \approx 10^{-8}$, $\lambda_c \approx 10^{-6}$ and $K \approx 0.09$, then,

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{10^{-8}}{10^{-6} \times 0.09} = \frac{1}{9} \quad \text{i.e,}$$

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| \ll 1 \quad \text{for any } \tau, \ 0 \leq \tau \leq \infty \qquad (4.50)$$

Hence, inequalities (4.49) and (4.50) lead to the following conclusion:

$$\left|\frac{\xi}{Kue^{K\tau}}\right| \ll 1 \quad \text{for any} \quad 0 \leq \tau \leq \infty \qquad (4.51)$$

This means that the nonlinear transformation (3.1) may be applicable in this case.

b. Case $K < \mu_o^2$

Inequality (4.47) can be rewritten as

$$|\eta_1'| < \frac{2\alpha}{K} B_o^2 \mu_o^2 e^{(K-2\mu_o^2)\tau}$$

and so,

$$\left|\frac{\eta_1'}{Ku e^{K\tau}}\right| < \frac{2\alpha}{K^2} B_o \mu_o^2 e^{-\mu_o^2 \tau} < \frac{2\alpha}{K^2} B_o \mu_o^2 \text{ for any } \tau \geq 0$$

Let $\mu_o^2 = 0.09$, $K \simeq 0.08$, $\alpha \simeq 10^{-21}$, $B_o \simeq 10^{10}$, λ_c 10^{-6}

then,

$$\left|\frac{\eta_1'}{Ku e^{K\tau}}\right| < \frac{2\times 10^{-21} \times 10^{10} \times 0.09}{(0.08)^2} \approx 3. \times 10^{-10}$$

or

$$\left|\frac{\eta_1'}{Ku e^{K\tau}}\right| \ll 1 \quad \text{for any } \tau \geq 0 \tag{4.52}$$

Similarly, inequality (4.48) turns out to be:

$$|\eta_2'| < \beta^* B_o e^{-K_o\tau} \int_0^\tau e^{(K_o-\mu_o^2)\tau'} d\tau' = \frac{\beta^* B_o}{K_o - \mu_o^2} e^{-K\tau} \left[e^{(K_o-\mu_o^2)\tau} - 1 \right] \tag{4.53}$$

Since, $K_o - \mu_o^2 = K + \lambda_c - \mu_o^2 \simeq 0.08 + 10^{-6} - 0.09 \simeq -0.01$

it can be assumed that

$$(K_o - \mu_o^2) < 0$$

then, inequality (4.53) yields:

$$|\eta_2'| < \frac{\beta^* B_o}{\mu_o^2 - K_o} e^{-K\tau} \left[1 - e^{(K_o-\mu_o^2)\tau} \right] < \frac{\beta^* B_o}{\mu_o^2 - K_o} e^{-K\tau}$$

or

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{\beta^*}{K(\mu_o^2-K_o)} e^{-(2K-\mu_o^2)\tau}$$

If $2K - \mu_o^2 \geq 0$, i.e., $K \geq \frac{\mu_o^2}{2}$, it is obtained from the above inequality that:

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{\beta^*}{K(\mu_o^2-K_o)} \qquad \text{for all } \tau \geq 0$$

or

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{10^{-8}}{0.08(0.01)} \approx \frac{1}{8} \times 10^{-4}$$

This implies that

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| \ll 1 \qquad \text{for all } \tau \geq 0 \tag{4.54}$$

Expressions (4.52) and (4.54) prove that the value of $\left|\frac{\xi(x,t)}{Kue^{KT}}\right|$ can be considered negligibly small compared to unity for any $0 \leq \tau \leq \infty$ if $\frac{\mu_o^2}{2} \leq K < \mu_o^2$.

c. Case $K > \mu_o^2$

It follows from inequality (4.47) that

$$|\eta_1'| < \frac{2\alpha B_o^2 \mu_o^2}{K} e^{(K-2\mu_o^2)\tau} \qquad \text{for any } \tau \geq 0$$

hence,

$$\left|\frac{\eta_1'}{Kue^{K\tau}}\right| < \frac{2\alpha B_o \mu_o^2}{K^2} e^{-\mu_o^2\tau} < \frac{2\alpha B_o \mu_o^2}{K^2} \qquad \text{for any } \tau \geq 0$$

Let $\alpha = 10^{-21}$, $B_o = 10^{10}$, $\mu_o^2 \simeq 0.09$, $K \simeq 0.093$

Then,

$$\left|\frac{\eta_1'}{K u e^{K\tau}}\right| < \frac{2\times10^{-21}\times10^{10}\times0.09}{(0.093)^2} = 2.1\times10^{-8}$$

or $$\left|\frac{\eta_1'}{K u e^{K\tau}}\right| \ll 1 \qquad \text{for any } \tau \geq 0 \qquad (4.55)$$

To prove the ratio $\dfrac{\eta_2'}{Kue^{K\tau}}$ is very small compared to one for any $\tau \geq 0$, it is divided in three different cases for the sake of simplicity:

1.- If $a_1 e^{(K-\mu_0^2)\tau} \ll 1$ i.e,

$$e^{(K-\mu_0^2)\tau} \ll \frac{1}{a_1} \approx \frac{K}{\alpha B_0}$$

or

$$\tau \ll \frac{1}{(K-\mu_0^2)} \ln\left(\frac{K}{\alpha B_0}\right) \approx \frac{1}{0.003} \ln\left(\frac{0.093}{10^{-21}\times10^{10}}\right)$$

$$\tau \ll 7667$$

Expression (4.48) yields the inequality:

$$|\eta_2'| < \beta^* B_0 e^{-K_0\tau} \int_0^{\tau} e^{(K_0-\mu_0^2)\tau'} d\tau'$$

or

$$|\eta_2'| < \frac{\beta^* B_0}{K_0-\mu_0^2} e^{-K_0\tau}\left[e^{(K_0-\mu_0^2)\tau} - 1\right] < \frac{\beta^* B_0}{K_0-\mu_0^2} e^{-\mu_0^2\tau}$$

and

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{\beta^*}{K(K_0-\mu_0^2)} e^{-K\tau} < \frac{\beta^*}{K(K_0-\mu_0^2)} \qquad \text{for any } \tau \geq 0$$

Assume $\beta^* = 10^{-8}$, $K = 0.093$, $K_0 - \mu_0^2 \simeq 0.003$

Then,

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{10^{-8}}{0.093\times0.005} \approx 4\times10^{-5} \ll 1 \quad \text{for any } \tau \geq 0 \qquad (4.56)$$

2.- If $a_1 e^{(K-\mu_0^2)\tau} \gg 1$ i.e,

$$\tau \gg \frac{1}{K-\mu_0^2} \ln\left(\frac{K}{\alpha B_0}\right) \approx 7667$$

It is obtained from inequality (4.48) that:

$$|\eta_2'| < \beta^* B_0 e^{-K_0\tau} a_1^2 e^{2(K-\mu_0^2)\tau} \int_0^\tau \frac{e^{(K_0-\mu_0^2)\tau'}}{a_1 e^{(K-\mu_0^2)\tau'}} d\tau'$$

$$< \beta^* B_0^2 \frac{\alpha}{K} e^{(-K_0+2K-2\mu_0^2)\tau} \int_0^\tau e^{\lambda_c \tau'} d\tau'$$

$$< \frac{\beta^* B_0^2 \alpha}{K\lambda_c} e^{(K-2\mu_0^2)\tau}$$

then,

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{\beta^* B_0 \alpha}{K^2\lambda_c} e^{-\mu_0^2\tau} < \frac{\beta^* B_0\alpha}{K^2\lambda_c} \quad \text{for any } \tau \geqslant 0$$

or

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{10^{-8}\times10^{10}\times10^{-21}}{(0.093)^2(10^{-6})} \approx 10^{-11} \ll 1 \quad \text{for any } \tau \qquad (4.57)$$

3.- If $a_1 e^{(K-\mu_0^2)\tau} \approx 1$ i.e,

$$\tau \approx \frac{1}{K-\mu_0^2} \ln\left(\frac{K}{\alpha B_0}\right) \approx 7667$$

then,

$$|\eta_2'| < 4\beta^* B_0 e^{-K_0\tau}\int_0^\tau e^{(K_0-\mu_0^2)\tau'}d\tau' < \frac{4\beta^* B_0 e^{-K_0\tau}}{(K_0-\mu_0^2)} e^{(K_0-\mu_0^2)\tau}$$

$$< \frac{4\beta^* B_0}{(K_0-\mu_0^2)} e^{-\mu_0^2\tau}$$

It is deduced from the above inequality that:

$$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{4\beta^*}{K(K_0-\mu_0^2)} e^{-K\tau} < \frac{4\beta^*}{K(K_0-\mu_0^2)} \quad \text{for any } \tau \geq 0$$

or $$\left|\frac{\eta_2'}{Kue^{K\tau}}\right| < \frac{4\times 10^{-8}}{0.093\times 0.003} \approx 10^{-4} \ll 1 \qquad (4.58)$$

Therefore, inequalities (4.55), (4.56), (4.57) and (4.58) shows that the conditions (3.13) and (3.14) are satisfied for any $\tau \geq 0$. In other words, for all cases, it can be written as follows:

$$\frac{\mathcal{E}}{Kue^{K\tau}} \ll 1 \qquad \text{for any } \tau \geq 0 \qquad (4.59)$$

C. CONVERGENCE OF THE SOLUTION

Consider the sequence of successive approximations

$$\{u_i(x,\tau)\,|\,u_i(x,\tau) \text{ is defined by equation (4.39)}\} \qquad (4.60)$$

$$u_i(x,\tau) = u_0(x,\tau) + \int_0^\tau \int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_i(\xi,t)\,d\xi\,dt \qquad (4.39)$$

Where $G(x,\xi,\tau-t) = \frac{1}{a}\sum_{n=0}^{\infty} e^{-\mu_n^2(\tau-t)}\cos\mu_n x\cos\mu_n\xi$

The Green's function $G(x,\xi,\tau)$ is continuous in $-a \leq x,\xi \leq a$ for $0 < \tau \leq \infty$ [cf. appendix A] and bounded. Thus, the convergence of the sequence $\{u_i(x,\tau)\}$ reduces to that of $\{\mathcal{E}_i(x,\tau)\}$ for $0 < \tau < \infty$.

Since the ratio $\dfrac{\mathcal{E}_i}{Ku_i e^{K\tau}}$ is very small compared to unity for $0 \leq \tau \leq \infty$, we have:

$$\mathcal{E}_i \ll K u_i e^{K\tau}$$

For $0 \leq \tau \leq \tau_1$, the function $u_i(x,\tau)$ is analytically well-behaved, hence bounded by a number M and the inequality above can be written as

$$\mathcal{E}_i \ll K M e^{K\tau_1}$$

which implies that the $\mathcal{E}_i$'s should be bounded by a quantity ε'

$$\mathcal{E}_i < \varepsilon' \quad \text{for} \quad 0 \leq \tau \leq \tau_1$$

In addition, equation (4.39) gives the following recurrence formula:

$$u_{i+1} - u_i = \int_0^\tau \int_{-a}^{a} G(x,\xi,\tau-t)\left[\mathcal{E}_{i+1}(\xi,t) - \mathcal{E}_i(\xi,t)\right] d\xi\, dt \qquad (4.61)$$

$$i = 0, 1, 2, \ldots$$

where the $\mathcal{E}_i$'s are evaluated from expression (4.2). Since α is of order of magnitude about 10^{-21} and β^* about 10^{-8} [cf. appendix B], the first term in the right hand side of equation (4.2) can be neglected compared to the second term, i.e.,

$$\mathcal{E}_i \approx \beta^*\left[1-\frac{\alpha}{K}(1-e^{K\tau})u_{i-1}\right]^2\int_0^{\tau}\frac{e^{-K_0(\tau-\tau')}u_{i-1}(x,\tau')\,d\tau'}{1-\frac{\alpha}{K}(1-e^{K\tau'})u_{i-1}(x,\tau')} \tag{4.62}$$

It can be seen from equation (4.62) that:

$$\mathcal{E}_i \geqslant 0$$

since the integrand under the integral is non-negative.

The zeroth approximation yields $\mathcal{E}_o = 0$ and $u_o(x,\tau)$. Next, equation (4.61) gives:

$$u_1 - u_o = \int_0^{\tau}\int_{-a}^{a} G(x,\xi,\tau-t)\,\mathcal{E}_1(\xi,t)\,d\xi\,dt$$

where $$G(x,\xi,\tau-t) \geqslant 0$$

and $$\mathcal{E}_1(\xi,t) \geqslant 0$$

Therefore, we obtain

$$u_1 - u_o \geqslant 0$$

or $$u_1 \geqslant u_o \tag{4.63}$$

The values of $\mathcal{E}_i$ corresponding to u_o and u_1 are given by equation (4.62):

$$\mathcal{E}_1 \approx \beta^*\left[1+\frac{\alpha}{K}(e^{K\tau}-1)u_o\right]^2\int_0^{\tau}\frac{e^{-K_0(\tau-\tau')}u_o(x,\tau')\,d\tau'}{1-\frac{\alpha}{K}(1-e^{K\tau'})u_o(x,\tau')} \tag{4.64}$$

$$\mathcal{E}_2 \simeq \beta^*\left[1+\frac{\alpha}{K}(e^{K\tau}-1)u_1\right]^2\int_0^{\tau}\frac{e^{-K_0(\tau-\tau')}u_1(x,\tau')\,d\tau'}{1-\frac{\alpha}{K}(1-e^{K\tau'})u_1(x,\tau')} \tag{4.65}$$

The values of u_o and u_1 do not much change the integrals in equations (4.64) and (4.65) but do change appreciably

the factors before the integrals. Furthermore, $u_1 \geq u_o$, hence,

$$\mathcal{E}_2 \geqslant \mathcal{E}_1 \tag{4.66}$$

Combining equation (4.61) and inequality (4.66) leads to the following inequality:

$$u_2 - u_1 \geqslant 0$$

or

$$u_2 \geqslant u_1 \tag{4.67}$$

Following the same procedure, it can be shown that:

$$\mathcal{E}_0 \leq \mathcal{E}_1 \leq \mathcal{E}_2 \leq \cdots \leq \mathcal{E}_i \leq \mathcal{E}_{i+1} \leq \cdots \tag{4.68}$$

and

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_i \leq u_{i+1} \leq \cdots \tag{4.69}$$

The sequence $\{\mathcal{E}_i\}$ is bounded and monotonic increasing, so it converges [11]. Therefore the convergence of the sequence $\{u_i\}$ is proved.

D. ASYMPTOTIC EQUILIBRIUM STATES FOR AN INFINITE SLAB REACTOR

Stability is one of fundamental problems in nuclear reactor design. Some questions are raised here. Is the system stable if the steady-state flux $\phi_o(x)$ is perturbed locally or if the generalized buckling K is changed by changes in the absorption cross section due to control-rod movement? So, this sub-section is devoted to the investigation of all asymptotic equilibrium states (if any) from the solution obtained in paragraph A of this section.

1. Zeroth Approximation

The zeroth approximation of the increase of neutron flux above its steady value is given by equations (4.18) and (4.40):

$$\psi_0(x,\tau) = \frac{\sum_{n=0}^{\infty} B_n e^{-(\mu_n^2 - K)\tau} \cos\mu_n x}{1 - \frac{\alpha}{K}\sum_{n=0}^{\infty} B_n \left[e^{-\mu_n^2\tau} - e^{-(\mu_n^2 - K)\tau}\right]\cos\mu_n x} \tag{4.70}$$

where the B_n's are defined by equation (4.20).

a. Case $K = \mu_o^2 = (\frac{\pi}{2a})^2$

All $(\mu_n^2 - K)$ are positive for $n > 0$ since

$$\mu_0 < \mu_1 < \cdots < \mu_n < \mu_{n+1} < \cdots$$

Therefore, all exponential terms except the term in $(\mu_o - K)$ will become negligibly small after sufficiently long time and equation (4.70) reduces to

$$\psi_0(x,\tau) \longrightarrow \frac{B_0 \cos\frac{\pi}{2a}x}{1 + \frac{\alpha}{K} B_0 \cos\frac{\pi}{2a}x}, \quad \text{as } \tau \to \infty \tag{4.71}$$

which describes the zeroth approximation for an asymptotic equilibrium state.

For the case of no delayed neutron ($\beta = 0$) and no feedback ($\alpha = 0$), expression (4.71) becomes:

$$\psi_0(x,\tau) \longrightarrow B_0 \cos\frac{\pi}{2a}x, \quad \text{as } \tau \to \infty \tag{4.72}$$

This steady state is exactly the same as that obtained in Reactor Analysis [6].

b. Case $\mu_o^2 < K < \mu_1^2$

All exponents $(\mu_n^2 - K)$ with $n > 0$ are positive, hence the corresponding terms in equation (4.100) decrease with time τ except the term in $(\mu_o^2 - K)$ which increases with time since $\mu_o^2 - K$ is negative. Thus, as τ goes to infinity, $\psi_o(x,\tau)$ approaches the following limit:

$$\psi_o(x,\tau) \longrightarrow \frac{B_o e^{-(\mu_o^2-K)\tau} \cos\mu_o x}{1 + \frac{\alpha}{K} B_o e^{-(\mu_o^2-K)\tau} \cos\mu_o x}$$

Since $\frac{\alpha}{K} B_o e^{-(\mu_o^2-K)\tau} \cos\mu_o x \gg 1$ as $\tau \to \infty$

the stable equilibrium is

$$\psi_o(x,\tau) \longrightarrow \frac{K}{\alpha} \tag{4.73}$$

For the linear case, the reactor would be called supercritical since the flux $\psi_o(x,\tau)$ will tend to infinity as seen in equation (4.73) by putting $\alpha = 0$. Again, this result is consistent with previous Reactor Analysis [6].

c. Case $K < \mu_o^2$

All exponential terms in equation (4.70) decrease to zero as time τ increases to infinity. Thus, the increase of neutron flux above its steady value is zero at the asymptotic equilibrium state for the nonlinear case.

$$\psi_o(x,\tau) \longrightarrow 0 \quad \text{as} \quad \tau \to \infty \tag{4.74}$$

This situation is called subcritical in a linear reactor.

2. <u>First Approximation</u>

The first approximation of the increase of neutron flux above its steady value is obtained by inserting equation (4.38) into equation (4.40) and putting i = 1:

$$\psi_1(x,\tau) = \frac{e^{\kappa\tau}u_0(x,\tau) + e^{\kappa\tau}\int_0^\tau\int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\,d\xi\,dt}{1-\frac{\alpha}{k}(1-e^{\kappa\tau})u_0(x,\tau) - \frac{\alpha}{k}(1-e^{\kappa\tau})\int_0^\tau\int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\,d\xi\,dt} \tag{4.75}$$

where $u_0(x,\tau)$, $G(x,\xi,\tau-t)$ and $\mathcal{E}_1(\xi,t)$ are defined by equations (4.18), (4.22) and equation (4.2) respectively. On the other hand, $\mathcal{E}_1(\xi,t)$ could be written in the form:

$$\mathcal{E}_1(\xi,t) = \eta_1'(\xi,t) + \eta_2'(\xi,t) \tag{4.76}$$

where $\eta_1'(\xi,t)$ and $\eta_2'(\xi,t)$ are determined from equations (4.41) and (4.42).

Only the fundamental mode is concerned with in the following calculations. Therefore, the values of $u_0(\xi,t)$, $\frac{\partial u_0}{\partial \xi}$ and $G(x,\xi,\tau-t)$ are given by:

$$u_0(\xi,t) = B_0 e^{-\mu_0^2 t}\cos\mu_0\xi \tag{4.77}$$

$$\frac{\partial u_0}{\partial \xi}(\xi,t) = -B_0\mu_0 e^{-\mu_0^2 t}\sin\mu_0\xi \tag{4.78}$$

$$G(x,\xi,\tau-t) = \frac{1}{a} e^{-\mu_0^2(\tau-t)}\cos\mu_0 x\cos\mu_0\xi \tag{4.79}$$

Let

$$I_1(x,\tau) = e^{\kappa\tau}\int_0^\tau\int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\,d\xi\,dt \tag{4.80}$$

$$I_2(x,\tau) = \int_0^\tau\int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\,d\xi\,dt \tag{4.81}$$

hence, the relationship between $I_1(x,\tau)$ and $I_2(x,\tau)$ is established,

$$I_1(x,\tau) = e^{K\tau} I_2(x,\tau) \tag{4.82}$$

and equation (4.75) becomes:

$$\psi_1(x,\tau) = \frac{e^{K\tau} u_0(x,\tau) + I_1(x,\tau)}{1 - \frac{\alpha}{K}(1-e^{K\tau}) u_0(x,\tau) - \frac{\alpha}{K} I_2(x,\tau) + \frac{\alpha}{K} I_1(x,\tau)} \tag{4.83}$$

Substituting equation (4.77) for $u_o(x,\tau)$ into equation (4.83) yields:

$$\psi_1(x,\tau) \qquad \frac{B_0 e^{-(\mu_0^2-K)\tau} \cos\mu_0 x + I_1}{1 - \frac{\alpha}{K} B_0 \left[e^{-\mu_0^2\tau} - e^{-(\mu_0^2-K)\tau} \right] \cos\mu_0 x - \frac{\alpha}{K} I_2 + \frac{\alpha}{K} I_1} \tag{4.84}$$

Taking the limit of the above equation, as τ tends to infinity, gives:

$$\lim_{\tau\to\infty} \psi_1(x,\tau) = \lim_{\tau\to\infty} \left\{ \frac{B_0 e^{-(\mu_0^2-K)\tau} \cos\mu_0 x + I_1}{1 - \frac{\alpha}{K} B_0 \left[e^{-\mu_0^2\tau} - e^{-(\mu_0^2-K)\tau} \right] \cos\mu_0 x - \frac{\alpha}{K} I_2 + \frac{\alpha}{K} I_1} \right\} \tag{4.85}$$

It is apparent that the temporal behavior of $\psi_1(x,\tau)$ depends on the generalized buckling K. The limits of I_1 and I_2, as τ approaches infinity, are evaluated as follows:

$$\lim_{\tau\to\infty} \int_0^\tau \int_{-a}^{a} G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\, d\xi\, dt = \lim_{\tau\to\infty} \int_{-a}^{a} \int_0^\tau G(x,\xi,\tau-t)\mathcal{E}_1(\xi,t)\, dt\, d\xi \tag{4.86}$$

Thus, the problem is reduced to finding the value of the integral,

$$\int_0^{\tau} G(x,\xi,\tau-t)\,\zeta_1(\xi,t)\,dt \quad \text{for } \tau \text{ large}$$

which is the sum of two integrals:

$$\int_0^{\tau} G(x,\xi,\tau-t)\,\eta_1'(\xi,t)\,dt \quad \text{and} \quad \int_0^{\tau} G(x,\xi,\tau-t)\,\eta_2'(\xi,t)\,dt$$

in view of equation (4.76).

Equations (4.41), (4.77), (4.78) and equation (4.79) give:

$$\int_0^{\tau} G(x,\xi,\tau-t)\eta_1'(\xi,t)\,dt = -\frac{2\alpha}{aK}B_0^2\mu_0^2 e^{-\mu_0^2\tau}\cos\mu_0 x\cos\mu_0\xi\sin^2\mu_0\xi\int_0^{\tau}\frac{e^{-(\mu_0^2-K)t}-e^{-\mu_0^2 t}}{1-\frac{\alpha}{K}B_0\left[e^{-\mu_0^2 t}-e^{-(\mu_0^2-K)t}\right]\cos\mu_0\xi}\,dt$$

Let $\quad A_1(x,\xi) = -\frac{2\alpha}{aK}B_0^2\mu_0^2\cos\mu_0 x\cos\mu_0\xi\sin^2\mu_0\xi \qquad (4.87)$

and $\quad a_1(\xi) = \frac{\alpha}{K}B_0\cos\mu_0\xi \qquad (4.88)$

The above equation can be written in the form:

$$\int_0^{\tau} G\eta_1'\,dt = A_1(x,\xi)e^{-\mu_0^2\tau}\int_0^{\tau}\frac{e^{-(\mu_0^2-K)t}-e^{-\mu_0^2 t}}{1-a_1(\xi)e^{-\mu_0^2 t}+a_1(\xi)e^{-(\mu_0^2-K)t}}\,dt \qquad (4.89)$$

Similarly, taking account of equations (4.42), (4.77), (4.78) and equation (4.79) yields:

$$\int_0^{\tau} G\eta_2'\,dt = \frac{\beta^* B_0}{a}e^{-\mu_0^2\tau}\cos\mu_0 x\cos^2\mu_0\xi\int_0^{\tau} e^{(\mu_0^2-K)t}\left[1-a_1e^{-\mu_0^2 t}+a_1e^{-(\mu_0^2-K)t}\right]^2\int_0^{t}\frac{e^{-(\mu_0^2-K_0)\tau'}\,d\tau'\,dt}{1-a_1e^{-\mu_0^2\tau'}+a_1e^{-(\mu_0^2-K)\tau'}},$$

Changing the order of integration in the above equation and letting

$$A_2(x,\xi) = \frac{\beta^*}{a} B_0 \cos\mu_0 x \cos^2\mu_0\xi \tag{4.90}$$

leads to the following equation:

$$\int_0^\tau G\eta_2' dt = A_2 e^{-\mu_0^2\tau} \int_0^\tau \frac{e^{-(\mu_0^2-K_0)\tau'}}{1-a_1 e^{-\mu_0^2\tau'} + a_1 e^{-(\mu_0^2-K)\tau'}} \int_{\tau'}^{\tau} e^{(\mu_0^2-K_0)t} \left[1-a_1 e^{-\mu_0^2 t} + a_1 e^{-(\mu_0^2-K)t}\right]^2 dt\, d\tau' \tag{4.91}$$

a. Case $K = \mu_o^2$

As t is sufficiently large, the integrand of the integral in the right hand side of equation (4.89) is equivalent to:

$$\frac{e^{-(\mu_0^2-K)t} - e^{-\mu_0^2 t}}{1-a_1 e^{-\mu_0^2 t} + a_1 e^{-(\mu_0^2-K)t}} \simeq \frac{1}{1+a_1(\xi)} \tag{4.92}$$

Substituting equation (4.92) into equation (4.89) gives:

$$\int_0^\tau G\eta_1' dt \simeq A_1(x,\xi)\, e^{-\mu_0^2\tau} \int_0^\tau \frac{dt}{1+a_1(\xi)}$$

or

$$\int_0^\tau G\eta_1' dt \simeq A_3(x,\xi)\, e^{-\mu_0^2\tau}\, \tau \tag{4.93}$$

where $$A_3(x,\xi) = \frac{A_1(x,\xi)}{1+a_1(\xi)} \tag{4.94}$$

In the same manner, equation (4.91) can be written as follows:

$$\int_0^\tau G\eta_2' \, dt \simeq A_2(x,\xi) e^{-\mu_0^2\tau} \int_0^\tau \frac{e^{-(\mu_0^2-\kappa_0)\tau'}}{1+a_1(\xi)} \int_{\tau'}^{\tau} (1+a_1)^2 e^{(\mu_0^2-\kappa_0)t} \, dt \, d\tau'$$

when τ is sufficiently large or after integrating:

$$\int_0^\tau G\eta_2' \, dt \simeq -\frac{A_2(1+a_1)}{\mu_0^2-\kappa_0}\left[\tau - \frac{e^{(\mu_0^2-\kappa_0)\tau}}{\mu_0^2-\kappa_0} + \frac{1}{\mu_0^2-\kappa_0}\right] e^{-\mu_0^2\tau} \tag{4.95}$$

But equation (3.8) is recalled:

$$\kappa_0 = \kappa + \lambda_c$$

or

$$\kappa_0 - \mu_0^2 = \kappa - \mu_0^2 + \lambda_c$$

since

$$\kappa - \mu_0^2 = 0$$

hence,

$$\kappa_0 - \mu_0^2 = \lambda_c \tag{4.96}$$

Inserting equation (4.96) into equation (4.95) and putting

$$A_4(x,\xi) = \frac{A_2(1+a_1)}{\lambda_c} \tag{4.97}$$

leads to the following result:

$$\int_0^\tau G\eta_2' \, dt \simeq A_4 e^{-\mu_0^2\tau}\left(\tau + \frac{e^{-\lambda_c\tau}}{\lambda_c} - \frac{1}{\lambda_c}\right)$$

or, as τ is sufficiently large:

$$\int_0^\tau G\eta_2' \, dt \simeq A_4 e^{-\mu_0^2\tau}\tau \tag{4.98}$$

Now, adding equation (4.93) to equation (4.98) yields:

$$\int_0^{\tau} G\xi_1\,dt = \int_0^{\tau} G\eta_1'\,dt + \int_0^{\tau} G\eta_2'\,dt \simeq (A_3+A_4)e^{-\mu_0^2\tau}\tau \qquad (4.99)$$

Integrating equation (4.99) with respect to ξ from -a to a will give the integral $I_2(x,\tau)$ as defined by equation (4.81)

$$I_2(x,\tau) = \int_{-a}^{a}\int_0^{\tau} G\xi_1\,dt\,d\xi \simeq A_5(x)e^{-\mu_0^2\tau}\tau \qquad (4.100)$$

where
$$A_5(x) = \int_{-a}^{a}[A_3(x,\xi)+A_4(x,\xi)]\,d\xi \qquad (4.101)$$

The integral $I_1(x,\tau)$ is evaluated by using relation (4.82) and equation (4.100)

$$I_1(x,\tau) = e^{K\tau} I_2(x,\tau) \simeq e^{K\tau}e^{-\mu_0^2\tau}A_5(x)\tau$$

Since $K = \mu_o^2$, then

$$I_1(x,\tau) \simeq A_5(x)\tau \qquad (4.102)$$

Replace equations (4.100) and (4.102) for $I_2(x,\tau)$ and $I_1(x,\tau)$ in equation (4.85) with $K = \mu_o^2$ to obtain:

$$\lim_{\tau\to\infty} \psi_1(x,\tau) = \lim_{\tau\to\infty}\left\{\frac{B_0\cos\mu_0 x + A_5(x)\tau}{1+a_1(x)-a_1(x)e^{-\mu_0^2\tau} - \frac{\alpha}{K}A_5(x)e^{-\mu_0^2\tau}\tau + \frac{\alpha}{K}A_5(x)\tau}\right\}$$

Since the term $A_5(x)\tau$ is much larger than the other terms in the right hand side of the above equation for τ large and $A_5(x)$ is always positive [cf. appendix B] for $-a < x < a$, the limit of $\psi_1(x,\tau)$ becomes:

$$\lim_{\tau\to\infty} \psi_1(x,\tau) = \lim_{\tau\to\infty}\left(\frac{A_5\tau}{\frac{\alpha}{K}A_5\tau}\right)$$

or $$\lim_{\tau \to \infty} \psi_1(x,\tau) = \frac{K}{\alpha} \tag{4.103}$$

i.e., $\psi_1(x,\infty) = \frac{K}{\alpha}$ is the first approximation for a stable equilibrium state when the generalized buckling K is equal to the lowest eigenvalue, μ_o^2, of equation (4.3).

b. Case $K > \mu_o^2$

If $(K - \mu_o^2)$ is positive, so also $(K_o - \mu_o^2)$ is, since $K_o = K + \lambda_c$.

For t large, the integrand under integral (4.89) becomes:

$$\frac{e^{-(\mu_o^2-K)t} - e^{-\mu_o^2 t}}{1-a_1(\xi)e^{-\mu_o^2 t} + a_1(\xi)e^{-(\mu_o^2-K)t}} \simeq \frac{e^{-(\mu_o^2-K)t}}{a_1(\xi)\,e^{-(\mu_o^2-K)t}} = \frac{1}{a_1(\xi)} \tag{4.104}$$

Similarly, for the integrands in equation (4.91)

$$\frac{e^{-(\mu_o^2-K_o)\tau'}}{1-a_1(\xi)e^{-\mu_o^2\tau'} + a_1(\xi)e^{-(\mu_o^2-K)\tau'}} \simeq \frac{e^{-(\mu_o^2-K_o)\tau'}}{a_1(\xi)\,e^{-(\mu_o^2-K)\tau'}} = \frac{1}{a_1(\xi)}\,e^{(K_o-K)\tau'}$$

$$\simeq \frac{1}{a_1(\xi)}\,e^{\lambda_c\tau'} \tag{4.105}$$

and

$$e^{(\mu_o^2-K_o)t}\left[1-a_1(\xi)e^{-\mu_o^2 t} + a_1(\xi)e^{-(\mu_o^2-K)t}\right]^2 \simeq e^{(\mu_o^2-K_o)t}\,a_1^2(\xi)e^{-2(\mu_o^2-K)t}$$

$$\simeq a_1^2(\xi)e^{(K-\mu_o^2-\lambda_c)t} \tag{4.106}$$

Substituting equation (4.104) back into equation (4.89) gives:

$$\int_0^\tau G\eta_1' \, dt \simeq A_1(x,\xi)\, e^{-\mu_0^2\tau}\int_0^\tau \frac{dt}{a_1(\xi)} = A_6(x,\xi)\, e^{-\mu_0^2\tau}\,\tau \tag{4.107}$$

where
$$A_6(x,\xi) = \frac{A_1(x,\xi)}{a_1(\xi)} \tag{4.108}$$

Inserting equations (4.105) and (4.106) into equation (4.91) yields the following equation:

$$\int_0^\tau G\eta_2' \, dt \simeq A_2\, e^{-\mu_0^2\tau}\int_0^\tau \frac{e^{\lambda_c\tau'}}{a_1(\xi)} \int_{\tau'}^\tau a_1(\xi)\, e^{(\kappa-\mu_0^2-\lambda_c)t} \, dt\, d\tau'$$

then, after integrating and some manipulations, it is obtained that:

$$\int_0^\tau G\eta_2' \, dt \simeq \frac{A_2 a_1}{(\kappa-\mu_0^2-\lambda_c)}\, e^{(\kappa-2\mu_0^2)\tau}\left[\frac{1-e^{-\lambda_c\tau}}{\lambda_c} - \frac{1-e^{-(\kappa-\mu_0^2)\tau}}{\kappa-\mu_0^2}\right] \tag{4.109}$$

The exponential terms in the bracket will approach zero as τ tends to infinity. Therefore, equation (4.109) turns out to be:

$$\int_0^\tau G\eta_2' \, dt \simeq \frac{A_2 a_1}{\kappa-\mu_0^2-\lambda_c}\, e^{(\kappa-2\mu_0^2)\tau}\left(\frac{1}{\lambda_c} - \frac{1}{\kappa-\mu_0^2}\right)$$

or
$$\int_0^\tau G\eta_2' \, dt \simeq \frac{A_2 a_1}{\lambda_c(\kappa-\mu_0^2)}\, e^{(\kappa-2\mu_0^2)\tau}$$

Let
$$A_7(x,\xi) = \frac{A_2(x,\xi)\, a_1(\xi)}{\lambda_c(\kappa-\mu_0^2)} \tag{4.110}$$

It follows that:

$$\int_0^\tau G\eta_2' \, dt \simeq A_7(x,\xi)\, e^{(\kappa-2\mu_0^2)\tau} \tag{4.111}$$

Again, the value of integral $\int_0^T G\mathcal{E}_1 dt$ is obtained by summing up equation (4.107) and equation (4.111):

$$\int_0^\tau G\mathcal{E}_1\, dt \simeq A_6(x,\xi) e^{-\mu_0^2\tau}\tau + A_7(x,\xi) e^{(K-2\mu_0^2)\tau}$$

$$\simeq e^{(K-2\mu_0^2)\tau}\left[A_6 e^{-(K-\mu_0^2)\tau}\tau + A_7\right] \qquad (4.112)$$

When τ is sufficiently large, the first term in the bracket is negligibly small, compared to the second term since $(K - \mu_0^2)$ is positive, hence

$$\int_0^\tau G\mathcal{E}_1\, dt \simeq A_7 e^{(K-2\mu_0^2)\tau} \qquad (4.113)$$

The value of $I_2(x,\tau)$ is deduced from integrating equation (4.113) with respect to ξ from -a to a:

$$I_2(x,\tau) = \int_{-a}^{a}\int_0^\tau G\mathcal{E}_1\, dt\, d\xi \simeq A_8(x) e^{(K-2\mu_0^2)\tau} \qquad (4.114)$$

Where $$A_8(x) = \int_{-a}^{a} A_7(x,\xi)\, d\xi \qquad (4.115)$$

$A_7(x,\tau)$ is defined by equations (4.110), (4.90) and equation (4.88).

The value of $I_1(x,\tau)$ is determined from relation (4.82)

$$I_1(x,\tau) \simeq A_8(x) e^{2(K-\mu_0^2)\tau} \qquad (4.116)$$

Inserting the values of $I_2(x,\tau)$ and $I_1(x,\tau)$ from equations (4.114) and (4.116) into equation (4.85) yields:

$$\lim_{\tau\to\infty} \psi_1(x,\tau) = \lim_{\tau\to\infty}\left\{\frac{B_o e^{(K-\mu_o^2)\tau}\cos\mu_o x + A_8(x) e^{2(K-\mu_o^2)\tau}}{1 - a_1(x)e^{-\mu_o^2\tau} + a_1(x)e^{(K-\mu_o^2)\tau} - \frac{\alpha}{K}A_8(x)e^{(K-2\mu_o^2)\tau} + \frac{\alpha}{K}A_8(x)e^{2(K-\mu_o^2)\tau}}\right\}$$

or

$$\lim_{\tau\to\infty} \psi_1(x,\tau) = \lim_{\tau\to\infty}\left\{\frac{e^{2(K-\mu_o^2)\tau}\left[B_o e^{-(K-\mu_o^2)\tau}\cos\mu_o x + A_8\right]}{e^{2(K-\mu_o^2)\tau}\left[e^{-2(K-\mu_o^2)\tau} - a_1 e^{-(2K-\mu_o^2)\tau} + a_1 e^{-(K-\mu_o^2)\tau} - \frac{\alpha}{K}A_8 e^{-K\tau} + \frac{\alpha}{K}A_8\right]}\right\}$$

$$= \lim_{\tau\to\infty}\left\{\frac{B_o e^{-(K-\mu_o^2)\tau}\cos\mu_o x + A_8}{e^{-2(K-\mu_o^2)\tau} - a_1 e^{-(2K-\mu_o^2)\tau} + a_1 e^{-(K-\mu_o^2)\tau} - \frac{\alpha}{K}A_8 e^{-K\tau} + \frac{\alpha}{K}A_8}\right\}$$

Since $(K - \mu_o^2)$ is positive, all exponential terms in the bracket go to zero as τ approaches infinity. On the other hand, $A_8(x)$ is always positive [cf. appendix B] for $-a < x < a$, thus, the limit of $\psi_1(x,\tau)$ for $K > \mu_o^2$ is:

$$\lim_{\tau\to\infty} \psi_1(x,\tau) = \frac{K}{\alpha} \qquad (4.117)$$

which describes the asymptotic equilibrium state of the increase of neutron flux, $\phi(x,\tau) - \phi_o(x)$. This is exactly the same result as obtained from equation (4.73) for the zeroth approximation.

c. Case $K < \mu_o^2$

The same procedure as carried out in parts (a) and (b), the stable equilibrium state, will be found by using equations (4.85), (4.89) and equation (4.91).

$$\frac{e^{-(\mu_o^2-K)t} - e^{-\mu_o^2 t}}{1 - a_1 e^{-\mu_o^2 t} + a_1 e^{-(\mu_o^2-K)t}} = \frac{1 - e^{-Kt}}{e^{(\mu_o^2-K)t} - a_1 e^{-Kt} + a_1} \qquad (4.118)$$

The above expression is obtained by dividing the numerator and the denominator of the left hand side by $e^{-(\mu_o^2-K)t}$.

Since $\mu_o^2 - K > 0$, equation (4.118) becomes for t large:

$$\frac{e^{-(\mu_o^2-K)t} - e^{-\mu_o^2 t}}{1 - a_1 e^{-\mu_o^2 t} + a_1 e^{-(\mu_o^2-K)t}} \simeq \frac{1}{e^{(\mu_o^2-K)t}} = e^{-(\mu_o^2-K)t} \tag{4.119}$$

hence, integral (4.119) can be evaluated as follows:

$$\int_0^\tau G\eta_1' \, dt \simeq A_1 e^{-\mu_o^2\tau}\int_0^\tau e^{-(\mu_o^2-K)t} dt = \frac{A_1 e^{-\mu_o^2\tau}}{\mu_o^2-K}\left[1 - e^{-(\mu_o^2-K)\tau}\right]$$

or

$$\int_0^\tau G\eta_1' \, dt \simeq A_9(x,\xi) e^{-\mu_o^2\tau} \tag{4.120}$$

where

$$A_9(x,\xi) = \frac{A_1(x,\xi)}{\mu_o^2-K} \tag{4.121}$$

Furthermore, integral (4.91) is equivalent to

$$\int_0^\tau G\eta_2' \, dt \simeq A_2 e^{-\mu_o^2\tau}\int_0^\tau e^{-(\mu_o^2-K_o)\tau'}\int_{\tau'}^\tau e^{(\mu_o^2-K_o)t} dt\, d\tau'$$

$$\simeq \frac{A_2 e^{-\mu_o^2\tau}}{\mu_o^2-K_o}\left[\frac{e^{(\mu_o^2-K_o)\tau}-1}{\mu_o^2-K_o} - \tau\right]$$

It is assumed that:

$$\mu_o^2 - K_o > 0$$

then, for τ large:

$$\int_0^{\tau} G\eta_2' \, dt \simeq \frac{A_2 e^{-\mu_0^2\tau}}{(\mu_0^2 - K_0)^2} e^{(\mu_0^2 - K_0)\tau}$$

$$\simeq A_{10}(x,\xi)\, e^{-K_0\tau} \qquad (4.122)$$

where $A_{10}(x,\xi) = \dfrac{A_2(x,\xi)}{(\mu_0^2 - K_0)^2}$ (4.123)

Adding equation (4.120) to equation (4.122) yields:

$$\int_0^{\tau} G\xi_1 \, dt \simeq A_9(x,\xi)e^{-\mu_0^2\tau} + A_{10}(x,\xi)e^{-K_0\tau}$$

Since $\mu_0^2 > K_0$

thus,

$$\int_0^{\tau} G\xi_1 \, dt \simeq A_{10}(x,\xi)e^{-K_0\tau} \qquad (4.124)$$

and

$$I_2(x,\tau) = \int_{-a}^{a}\int_0^{\tau} G\xi_1 \, dt \, d\xi \simeq A_{11}(x)e^{-K_0\tau} \qquad (4.125)$$

where $A_{11}(x) = \int_{-a}^{a} A_{10}(x,\xi)\, d\xi$ (4.126)

and

$$I_1(x,\tau) = e^{K\tau} I_2(x,\tau) \simeq A_{11}(x)e^{(K-K_0)\tau} = A_{11}(x)e^{-\lambda_c\tau} \qquad (4.127)$$

Finally, inserting the values of $I_2(x,\tau)$ and $I_1(x,\tau)$ from equation (4.125) and (4.127) into equation (4.85) yields the first approximation of the asymptotic equilibrium state:

$$\lim_{\tau \to \infty} \psi_1(x,\tau) = \lim_{\tau \to \infty} \left\{ \frac{B_o e^{-(\mu_o^2 - K)\tau} \cos \mu_o x + A_{11}(x) e^{-\lambda_c \tau}}{1 - a_1(x) e^{-\mu_o^2 \tau} + a_1(x) e^{-(\mu_o^2 - K)\tau} - \frac{\alpha}{K} A_{11}(x) e^{-K_o \tau} + \frac{\alpha}{K} A_{11}(x) e^{-\lambda_c \tau}} \right\}$$

Since all exponential terms decrease as τ increases, then:

$$\lim_{\tau \to \infty} \psi_1(x,\tau) = 0 \tag{4.128}$$

Thus, the same asymptotic state as for the zeroth approximation is obtained. The same conclusion can be made even if K_o is assumed to be greater than μ_o^2, instead of $K_o < \mu_o^2$.

E. PHYSICAL INTERPRETATION OF RESULTS

It can be seen from the zeroth and first approximation that higher approximations are not required for asymptotic equilibrium states in a nonlinear system with negative prompt feedback and one-group delayed neutron. The stable equilibrium states in the first approximation are verified by the results of Kastenberg and Chambre's stability analysis [1]. Furthermore, in the absence of feedback, the results obtained here reduce to those of the linear reactor analysis as mentioned in part D-1.

The equilibrium states are established in a nonlinear nuclear reactor at different levels which depend upon the value of the generalized buckling K. The results are interpreted as follows.

For $\tau < 0$, the reactor was in steady-state with flux $\phi_o(x)$.

At $\tau = 0$, the flux is raised to an amount of $F(x)$ from an external source. What happens to the flux distribution after some time ($\tau > 0$)?

1. If $K \geq (\frac{\pi}{2a})^2$, the neutron flux will reach the stable equilibrium state of value $\phi(x) = \phi_o(x) + \frac{K}{\alpha}$ after the transient given by equations (4.39) and (4.40) has subsided.

2. If $K < (\frac{\pi}{2a})^2$, the neutron flux will return to the initial steady value $\phi_o(x)$ as equilibrium state.

On the other hand, the equilibrium state is weakly dependent of delayed neutrons since the fraction of delayed neutrons (β) is very small compared to unity in the expression of parameter α:

$$\alpha = |(1-\beta)\alpha_0 \gamma_2|$$

If delayed neutrons are neglected ($\beta^* = 0$), equation (2.40) becomes

$$\frac{\partial \psi}{\partial \tau}(x,\tau) = \frac{\partial^2 \psi}{\partial x^2}(x,\tau) + K\psi(x,\tau) - \alpha\psi^2(x,\tau) \qquad (4.129)$$

For $K > (\frac{\pi}{2a})^2$, theorme I of Kastenberg and Chambré [1] gives the asymptotic stable equilibrium state of the system (4.129), which is the positive (nontrivial) solution of equation:

$$\lim_{\tau \to 0} \left[\frac{\partial^2 \psi}{\partial x^2}(x,\tau) + K\psi(x,\tau) - \alpha\psi^2(x,\tau) \right] = 0 \qquad (4.130)$$

It is apparent that $\psi(x) = \frac{K}{\alpha}$ satisfies this equation, thus is a positive solution to equation (4.130). Therefore, the

result obtained in part 1) is verified by this theorem for $K > (\frac{\pi}{2a})^2$. Similarly, theorem II of Kastenberg and Chambré shows that the equilibrium state for $K < (\frac{\pi}{2a})^2$ is $\psi(x) = 0$; i.e., the neutron flux returns to the initial steady state $\phi_o(x)$. This is again consistent with the result in part 2).

Theorem III states that if $K = \mu_o^2$, the asymptotical stable equilibrium of the system above is also the zero state ($\psi(x) = 0$). This completely differs from the present result as described in part 1). From the physical point of view, if the reactor eigenvalue is not changed, then the perturbed flux should eventually return to the initial equilibrium state $\psi(x) = 0$. The fact that for $K = \mu_o^2$, the first approximation contains a secular term at large τ [cf. equation (4.102] indicates that this is a special case for which the Wilhelm's transformation may not be valid. To draw a definite conclusion at this point, it appears that the first approximation of the solution for the case $K = \mu_o^2$ requires further considerations.

V. ANALYSIS OF A FINITE CYLINDRICAL REACTOR CORE

A. SOLUTION

Another geometric configuration which is of practical importance in Reactor Design is the cylinder of finite length. Consider therefore a bare cylinder of dimensionless extrapolated radius R and height 2ℓ, as shown in Fig. 5.1, with the z direction along the axis of the cylinder and the origin of the coordinate system located at the center. The flux in this reactor depends upon r,z (cylindrical coordinates) and τ. The reactor equation is now [cf. equation (2.40)]

$$\frac{\partial\psi}{\partial\tau}(r,z,\tau) = \nabla^2\psi + K\psi - \alpha\psi^2 + \beta^*\int_0^\tau e^{-\lambda_c(\tau-\tau')}\psi(r,z,\tau')\,d\tau' \tag{5.1}$$

for $\quad 0 \leq r < R$

where

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2} \tag{5.2}$$

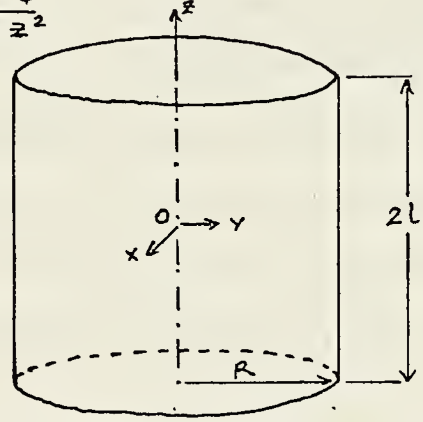

Figure 5.1. A Bare Cylindrical Reactor.

The initial and boundary conditions are:

$$
\begin{aligned}
\psi(r, z, 0) &= F(r, z) \\
\psi(R, z, \tau) &= 0 \\
\psi(0, z, \tau) &\text{ is finite} \\
\psi(r, \pm \ell, \tau) &= 0
\end{aligned}
\tag{5.3}
$$

Using the same transformation (3.1), equations (5.1) and (5.3) are transformed to linear equations as follows [cf. equation (3.31)]:

$$
\frac{\partial u_i(r,z,\tau)}{\partial \tau} = \nabla^2 u_i(r,z,\tau) + \mathcal{E}_i(r,z,\tau) \tag{5.4}
$$

$$
\begin{aligned}
u_i(r, z, 0) &= F(r, z) \\
u_i(R, z, \tau) &= 0 \\
u_i(0, z, \tau) &\text{ is finite} \\
u_i(r, \pm \ell, \tau) &= 0
\end{aligned}
\tag{5.5}
$$

where
$$
\mathcal{E}_0 = 0 \tag{5.6}
$$

and

$$
\mathcal{E}_i = -\frac{2\alpha(e^{K\tau}-1)\left[\left(\frac{\partial u_{i-1}}{\partial r}\right)^2+\left(\frac{\partial u_{i-1}}{\partial z}\right)^2\right]}{K\left[1-\frac{\alpha}{K}(1-e^{K\tau})u_{i-1}\right]} + \beta^*\left[1-\frac{\alpha}{K}(1-e^{K\tau})u_{i-1}\right]^2 \int_0^{\tau} \frac{e^{-K_0(\tau-\tau')}u_{i-1}(r,z,\tau')\,d\tau'}{1-\frac{\alpha}{K}(1-e^{K\tau'})u_{i-1}(r,z,\tau')} \tag{5.7}
$$

The methods used in the analysis of the infinite slab are easily extended to the analysis of the finite-cylinder reactor. The procedure is straight forward and involves the use of the technique of successive approximations and separation of variables.

1. Zeroth Approximation ($\mathcal{E}_o = 0$)

The appropriate form of the transformed equation which gives the zeroth approximation is:

$$\frac{\partial u_o}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_o}{\partial r}\right) + \frac{\partial^2 u_o}{\partial z^2} \tag{5.8}$$

$$\begin{aligned} u_o(r, z, 0) &= F(r, z) \\ u_o(R, z, \tau) &= 0 \\ u_o(0, z, \tau) &\text{ is finite} \\ u_o(r, \pm l, \tau) &= 0 \end{aligned} \tag{5.9}$$

A solution of this equation is assumed to be in the form:

$$u_o(r, z, \tau) = X(r)\, Z(z)\, T(\tau) \tag{5.10}$$

The substitution of this expression into equation (5.8) yields:

$$\frac{T'(\tau)}{T(\tau)} - \frac{1}{rX(r)}\frac{d}{dr}(rX') = \frac{Z''(z)}{Z(z)} = -\mu^2 \tag{5.11}$$

where μ is a constant to be determined from the boundary condition (5.9) or

$$Z(-l) = Z(l) = 0 \tag{5.12}$$

It follows from equation (5.11) that:

$$Z'' + \mu^2 Z = 0 \tag{5.13}$$

and

$$\frac{T'}{T} + \mu^2 = \frac{1}{r X} \frac{d}{dr}\left(r X'\right) = -\gamma^2 \qquad (5.14)$$

γ is another eigenvalue to be determined from the boundary conditions.

$$X(R) = 0 \qquad (5.15)$$

$$X(0) \text{ is finite}$$

Equation (5.14) can be written in 2 different equations:

$$\frac{d}{dr}\left(r X'\right) + \gamma^2 r X = 0 \qquad (5.16)$$

and

$$T' + (\mu^2 + \gamma^2) T = 0 \qquad (5.17)$$

Solving equations (5.12) and (5.13) gives:

$$\mu_n = (2n+1)\frac{\pi}{2l} \; ; \; n = 0, 1, 2, \cdots \qquad (5.18)$$

and

$$Z(z) = C_1 \cos \mu_n z \qquad (5.19)$$

Equation (5.16) is a Bessel's equation, the solution of which is of the form:

$$X(r) = C J_0(\gamma r) + D Y_0(\gamma r) \qquad (5.20)$$

where C,D are 2 constants

$$X(0) = C J_0(0) + D Y_0(0)$$

hence, D = 0 since $Y_0(0) = \infty$ and X(0) should be finite [cf. equation (5.15)]. Furthermore, the boundary condition X(R) = 0, yields:

$$J_0(\vartheta R) = 0 \tag{5.21}$$

If $\vartheta_0, \vartheta_1, \vartheta_2, \ldots \vartheta_m, \ldots$ are the positive roots of equation (5.21), the solutions to equation (5.16) are given by equation (5.20)

$$X(r) = cJ_0(\vartheta_m r) \tag{5.22}$$

and the solutions to equation (5.17) are

$$T(r) = E\, e^{-(\vartheta_m^2 + \mu_n^2)\tau} \tag{5.23}$$

where E is any constant.

Combining the product of solutions (5.19), (5.22) and (5.23) leads to the solution of equation (5.8)

$$u_0(r,z,\tau) = \sum_{m,n=0}^{\infty} A_{m,n} e^{-(\vartheta_m^2 + \mu_n^2)\tau} J_0(\vartheta_m r)\cos\mu_n z \tag{5.24}$$

The constants $A_{m,n}$ are defined by the initial condition

$$u_0(r,z,0) = F(r,z) = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} A_{m,n} J_0(\vartheta_m r)\right]\cos\mu_n z$$

This equation implies that:

$$\sum_{m=0}^{\infty} A_{m,n} J_0(\vartheta_m r) = \frac{1}{\ell}\int_{-\ell}^{\ell} F(r,\xi)\cos\mu_n\xi\, d\xi = F_n(r) \tag{5.25}$$

$$n = 0, 1, 2, \ldots .$$

Expression (5.25) is known as the Fourier-Bessel series representation of $F_n(r)$. Since γ_m are the positive roots of equation (5.21), the coefficients $A_{m,n}$ should have the form [12]

$$A_{m,n} = \frac{2}{R^2 J_1^2(\gamma_m R)} \int_0^R \eta J_0(\gamma_m \eta) F_n(\eta)\, d\eta$$

or

$$A_{m,n} = \frac{2}{\ell R^2 J_1^2(\gamma_m R)} \int_0^R \int_{-\ell}^{\ell} \eta J_0(\gamma_m \eta) F(\eta, \xi) \cos\mu_n \xi \, d\xi\, d\eta \qquad (5.26)$$

Substitution of expression (5.26) into equation (5.24) gives the solution $u_o(r,z,\tau)$:

$$u_o(r,z,\tau) = \int_0^R \int_{-\ell}^{\ell} \left[F(\eta,\xi) \frac{2\eta}{\ell R^2} \sum_{m,n=0}^{\infty} e^{-(\gamma_m^2+\mu_n^2)\tau} \frac{J_0(\gamma_m r)}{J_1^2(\gamma_m R)} J_0(\gamma_m \eta) \cos\mu_n z \cos\mu_n \xi \right] d\xi\, d\eta \qquad (5.27)$$

Let

$$G(r,z,\eta,\xi,\tau) = \frac{2\eta}{\ell R^2} \sum_{m,n=0}^{\infty} e^{-(\gamma_m^2+\mu_n^2)\tau} \frac{J_0(\gamma_m r)}{J_1^2(\gamma_m R)} J_0(\gamma_m \eta) \cos\mu_n z \cos\mu_n \xi \qquad (5.28)$$

$G(r,z,\eta,\xi,\tau)$ is called the Green's function of equation (5.8). Then, equation (5.27) becomes

$$\boxed{u_o(r,z,\tau) \quad \int_0^R \int_{-\ell}^{\ell} G(r,z,\eta,\xi,\tau) F(\eta,\xi)\, d\xi\, d\eta} \qquad (5.29)$$

2. <u>First Approximation</u>

The first approximation satisfies the following equation:

$$\frac{\partial u_1}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right) + \frac{\partial^2 u_1}{\partial z^2} + \mathcal{E}_1(r,z,\tau) \quad (5.30)$$

$$u_1(r,z,0) = F(r,z)$$

$$u_1(R,z,\tau) = 0$$

$$u_1(0,z,\tau) \text{ is finite} \quad (5.31)$$

$$u_1(r,\pm \ell,\tau) = 0$$

where $\mathcal{E}_1(r,z,\tau)$ is determined from equation (5.7) by making $i = 1$.

The solution of equation (5.20) is the sum of the solution u_o to equation (5.8) and the solution v to equations:

$$\frac{\partial v}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} + \mathcal{E}_1 \quad (5.32)$$

$$v(r,z,0) = 0$$

$$v(R,z,\tau) = 0$$

$$v(0,z,\tau) \text{ is finite} \quad (5.33)$$

$$v(r,\pm\ell,\tau) = 0$$

$$u_1 = u_o + v \quad (5.34)$$

The general solution to equatoin (5.32) is of the form:

$$v(r,z,\tau) = \sum_{m,n=0}^{\infty} B_{m,n}(\tau)\, J_0(\vartheta_m r) \cos \mu_n z \quad (5.35)$$

Expanding $\mathcal{E}_1$ in terms of $J_o(\nu_m r)\cos\mu_n z$ yields:

$$\mathcal{E}_1(r,z,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{m,n}(\tau) J_0(\nu_m r)\cos\mu_n z \tag{5.36}$$

where $\sum\limits_{m=0}^{\infty} C_{m,n} J_o(\nu_m r)$ are the Fourier's coefficients of the above series, i.e.,

$$\sum_{m=0}^{\infty} C_{m,n}(\tau) J_0(\nu_m r) = \frac{1}{\ell}\int_{-\ell}^{\ell} \mathcal{E}_1(r,\xi,\tau)\cos\mu_n\xi\, d\xi$$

This equation implies that $C_{m,n}(\tau)$ are the Bessel's series coefficients:

$$C_{m,n}(\tau) = \frac{2}{\ell R^2 J_1^2(\nu_m R)}\int_0^R\int_{-\ell}^{\ell} \eta J_0(\nu_m\eta)\mathcal{E}_1(\eta,\xi,\tau)\cos\mu_n\xi\, d\xi\, d\eta \tag{5.37}$$

Taking derivatives of equation (5.35) gives:

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \sum_{m,n=0}^{\infty} \frac{dB_{m,n}}{d\tau} J_0(\nu_m r)\cos\mu_n z \\
\frac{\partial v}{\partial r} &= -\sum_{m,n=0}^{\infty} \nu_m B_{m,n} J_1(\nu_m r)\cos\mu_n z \\
\frac{\partial^2 v}{\partial r^2} &= -\sum_{m,n=0}^{\infty} \nu_m^2 B_{m,n}\left[J_0(\nu_m r) - \frac{1}{\nu_m r}J_1(\nu_m r)\right]\cos\mu_n z \\
\frac{\partial^2 v}{\partial z^2} &= -\sum_{m,n=0}^{\infty} \mu_n^2 B_{m,n} J_0(\nu_m r)\cos\mu_n z
\end{aligned} \tag{5.38}$$

Insert equations (5.38) and (5.36) into equation (5.32) to obtain:

$$\sum_{m,n=0}^{\infty}\left[\frac{dB_{m,n}}{d\tau} + (\nu_m^2+\mu_n^2)B_{m,n} - C_{m,n}\right]J_0(\nu_m r)\cos\mu_n z = 0$$

This equation is true for any $0 \leq r < R$ and $-\ell < z < \ell$ if and only if:

$$\frac{dB_{m,n}}{d\tau} + (\nu_m^2+\mu_n^2)B_{m,n} = C_{m,n}(\tau) \tag{5.39}$$

The initial conditions are given by:

$$v(r,z,0) = 0 = \sum_{m,n=0}^{\infty} B_{m,n}(0)J_0(\nu_m r)\cos\mu_n z$$

or

$$B_{m,n}(0) = 0 \tag{5.40}$$

The solution to equation (5.39) can be written as:

$$B_{m,n}(\tau) = D_{m,n}e^{-(\nu_m^2+\mu_n^2)\tau} + e^{-(\nu_m^2+\mu_n^2)\tau}\int_0^{\tau} C_{m,n}(t)e^{(\nu_m^2+\mu_n^2)t}\,dt$$

where $D_{m,n}$ are constants determined from the initial conditions (5.40)

$$B_{m,n}(0) = D_{m,n} = 0$$

Therefore, it is deduced from the above equations that

$$B_{m,n}(\tau) \qquad \int_0^{\tau} C_{m,n}(t)\,e^{-(\nu_m^2+\mu_n^2)(\tau-t)}\,dt$$

Replacing equation (5.37) for $C_{m,n}(t)$ leads to

$$B_{m,n}(\tau) = \frac{2}{\ell R^2 J_1^2(\nu_m R)} \int_0^\tau \int_0^R \int_{-\ell}^{\ell} \eta J_0(\nu_m \eta)\, \mathcal{E}_1(\eta,\xi,t)\, e^{-(\nu_m^2+\mu_n^2)(\tau-t)} \cos\mu_n\xi \, d\xi\, d\eta\, dt \qquad (5.41)$$

Then, it follows from equations (5.35) and (5.41) that

$$v(r,z,\tau) = \int_0^\tau \int_0^R \int_{-\ell}^{\ell} \mathcal{E}_1(\eta,\xi,t)\, \frac{2\eta}{\ell R^2} \sum_{m,n=0}^{\infty} e^{-(\nu_m^2+\mu_n^2)(\tau-t)} \frac{J_0(\nu_m r)}{J_1^2(\nu_m R)} J_0(\nu_m \eta) \cos\mu_n z \cos\mu_n\xi \, d\xi\, d\eta\, dt$$

or, in terms of the Green's function (5.28), the above expression becomes:

$$v(r,z,\tau) \quad \int_0^\tau \int_0^R \int_{-\ell}^{\ell} \mathcal{E}_1(\eta,\xi,t)\, G(r,z,\eta,\xi,\tau-t)\, d\xi\, d\eta\, dt \qquad (5.42)$$

It is noted that this form of solution is analogous to that one derived in equation (4.37) for a slab reactor.

Now, combining equations (5.29) and (5.42) yields the first approximation:

$$\boxed{u_1(r,z,\tau) \quad u_0(r,z,\tau) + \int_0^\tau \int_0^R \int_{-\ell}^{\ell} G(r,z,\eta,\xi,\tau-t)\, \mathcal{E}_1(\eta,\xi,t)\, d\xi\, d\eta\, dt} \qquad (5.43)$$

3. ith Approximation

Since equation (5.4) is similar to equation (5.30) with the same initial boundary conditions, the ith approximation which is the solution to equation (5.4), can be obtained from expression (5.43) by substituting $\mathcal{E}_i(\eta,\xi,t)$ for $\mathcal{E}_1(\eta,\xi,t)$:

$$\boxed{u_i(r,z,\tau) \quad u_0(r,z,\tau) + \int_0^{\tau}\int_0^{R}\int_{-\ell}^{+\ell} G(r,z,\eta,\xi,\tau-t)\,\mathcal{E}_i(\eta,\xi,t)\,d\xi\, d\eta\, dt} \qquad (5.44)$$

Where $\mathcal{E}_i(\eta,\xi,t)$ is defined by equation (5.7). Equations (5.44) and (5.7) will give all possible successive approximations of the solution to equation (5.4) provided that $\mathcal{E}_o = 0$.

Finally, the ith approximation of the increase of neutron flux above its steady value is calculated from the transformation (3.1)

$$\psi_i(r,z,\tau) = \frac{e^{K\tau} u_i(r,z,\tau)}{1 - \frac{\alpha}{K}(1-e^{K\tau})\, u_i(r,z,\tau)} \qquad (5.45)$$

Thus, the analytical solution for a finite-cylinder reactor core with negative prompt feedback and one-group delayed neutron is established. The convergence of the series $\{u_i(r,z,\tau)\}$ can be shown by the same method as indicated in part IV-C for a slab reactor.

B. DISCUSSION OF RESULTS

From the solution (5.44) and the transformation (5.45), the asymptotic stable equilibrium states can be found at different levels by the same approach as applied in a slab reactor. The proof is not repeated here but only results are summarized as follows.

a. If $K = \mu_o^2 + \nu_o^2$

After a long time, the increase in neutron flux above its steady value will be constant with time and reach the value:

$$\psi_0(r,z,\tau) = \frac{A_0 J_0(\nu_0 r)\cos\mu_0 z}{1+\frac{\alpha}{K}A_0 J_0(\nu_0 r)\cos\mu_0 z} \tag{5.46}$$

where A_0 is given by equation (5.26)

$$A_0 = \frac{2}{\ell R^2 J_1^2(\nu_0 R)}\int_0^R \int_{-\ell}^{\ell} \eta J_0(\nu_0 \eta) F(\eta,\xi)\cos\mu_n \xi \, d\xi \, d\eta$$

b. If $K > \mu_0^2 + \nu_0^2$

The zeroth approximation for asymptotic equilibrium state is:

$$\psi_0(r,z,\infty) = \frac{K}{\alpha} \tag{5.47}$$

c. If $K < \mu_0^2 + \nu_0^2$

$$\psi_0(r,z,\infty) = 0 \tag{5.48}$$

It means that after the transient the neutron flux will return to the initial steady value as the stable equilibrium state. It is noted that in the absence of delayed neutron ($\beta = 0$) and feedback ($\alpha = 0$), equation (5.46) becomes:

$$\psi_0(r,z,\infty) = A_0 J_0(\nu_0 r)\cos\mu_0 z \tag{5.49}$$

for $K = \mu_0^2 + \nu_0^2$.

This result is exactly the same as that one obtained in previous work [6].

2. First Approximation

a. If $K \geq \mu_0^2 + \nu_0^2$

The neutron flux will approach the stable equilibrium value:

$$\phi(r, z, \infty) = \phi_0(r, z) + \frac{K}{\alpha} \tag{5.50}$$

as time goes to infinity.

b. If $K < \mu_o^2 + \nu_o^2$

The equilibrium state of neutron flux is the initial steady value $\phi_o(r,z)$

$$\phi(r, z, \infty) = \phi_0(r, z) \tag{5.51}$$

Therefore, the same physical interpretation can be made for a finite cylinder-reactor core as for an infinite slab-reactor core. It seems that the asymptotic states for a nonlinear nuclear reactor do not depend upon geometric configurations. This is true since the transformation is independent of coordinate systems and the transformed equation is maintained in the same form.

VI. CONCLUSIONS AND RECOMMENDATIONS

The objective of this thesis has been to provide an approach for directly solving the nonlinear space-time reactor kinetics equations. The theory is developed for an one-velocity, bare, homogeneous, one-group delayed neutron and Neutonian cooling system where the temperature rises instantaneously with the neutron flux and the temperature coefficient is negative (negative prompt feedback model).

The fundamental tool in this analysis is the Wilhelm's nonlinear transformation [8] that transforms a nonlinear equation to a weakly nonlinear one in which the nonlinear term represents a small perturbation. Then, the reduction of the later equation to a linear, parabolic type is established by any method of approximation. The successive approximations technique has been used in this thesis since it yields a general form of the ith approximation for the solution and it is easy to simulate the problem by digital computers if need be.

Furthermore, this transformation is independent of the system of coordinates. In other words, the transformed equation is maintained in a general form whatever coordinates are chosen. Therefore, the same conclusions of stability can be drawn from the resulting solutions for reactors of different geometric configurations. The asymptotic stable equilibrium states are governed by the relationship of the

generalized buckling K to μ_o^2 (or $\mu_o^2 + \gamma_o^2$), the lowest eigenvalue of the associated linear Helmholtz's equation. The value, $\phi_o(x) + \frac{K}{\alpha}$, is the stable state of neutron flux when $K \geq \mu_o^2$ (or $\mu_o^2 + \gamma_o^2$). When $K < \mu_o^2$, the neutron flux returns to its initial steady value $\phi_o(x)$ after some transient. Then, these results are compared with those derived from Theorems I, II and III of Kastenberg and Chambré [1] in the case of no-delayed neutron. As mentioned in part IV-E, the only inconsistency of the present work with Kastenberg and Chambré's is the stable equilibrium state for $K = \mu_o^2$. But if the comparison is made with linear reactor analysis (no delayed neutron and no feedback), the zeroth approximation (cf. see part IV-D-1) will give an asymptotic stable value which matches very well with the previous result [6]. So, for future work, it is recommended that a verification of the first approximation at $K = \mu_o^2$ by another approach, say digital computers, be made to insure its consistency with the theorem III of Kastenberg and with the physical point of view.

It is noted that the successive approximations method can be applied directly to equation (2.40) to obtain the solution provided that the coefficients α and β^* should be very small compared to other terms. On the other hand, the Wilhelm's transformation reduces equation (2.40) to equation (3.10) where the terms η_2 and η_2 become smaller, even if α and β^* are not appreciably small.

Finally, it is assumed from the beginning of this work, that the feedback coefficient α_o is negative. May the

transformation work with a positive feedback system ($\alpha_o > 0$ or $\alpha < 0$)? Moreover, if $\psi \geq 0$, $u(x,\tau)$ should be non-negative and less than the value, $-\dfrac{K}{\alpha(e^{K\tau}-1)}$. Otherwise, the function $\psi(x,\tau)$ will be negative, i.e., the range of $u(x,\tau)$ is limited in this case

$$0 \leq u < -\frac{K}{\alpha(e^{K\tau}-1)}, \quad \alpha < 0$$

and the singularity may occur when $u(x,\tau)$ is equal to $-\dfrac{K}{\alpha(e^{K\tau}-1)}$ at any point (x,τ). Therefore, it is not convenient to apply this transformation to a problem with positive feedback.

APPENDIX A

UNIFORM CONVERGENCE OF THE SERIES $\sum_{n=0}^{\infty} e^{-\mu_n^2\tau} \cos\mu_n x \cos\mu_n \xi$

Consider the above series where $\mu_n = (2n + 1)\frac{\pi}{2a}$. We will prove that this series converges uniformly for $-a \leq x, \xi \leq a$ and $0 < \tau_0 \leq \tau \leq \infty$.

$$(1) \quad \left| e^{-\mu_n^2\tau} \cos\mu_n x \cos\mu_n \xi \right| \leq e^{-\mu_n^2\tau} \leq e^{-\mu_n^2\tau_0} \quad \text{for} \quad \begin{cases} -a \leq x, \xi \leq a \\ \tau_0 \leq \tau \leq \infty \end{cases}$$

Expanding $e^{\mu_n^2\tau_0}$ into Taylor's series yields:

$$e^{\mu_n^2\tau_0} = 1 + \mu_n^2\tau_0 + \frac{1}{2!}(\mu_n^2\tau_0)^2 + \dots$$

hence,
$$e^{\mu_n^2\tau_0} \geq \mu_n^2\tau_0$$

and
$$e^{-\mu_n^2\tau_0} \leq \frac{1}{\mu_n^2\tau_0} = \frac{4a^2}{\pi^2\tau_0(2n+1)^2}$$

Therefore, inequality (1) can be written as:

$$(2) \quad \left| e^{-\mu_n^2\tau} \cos\mu_n x \cos\mu_n \xi \right| \leq \frac{4a^2}{\pi^2\tau_0^2} \frac{1}{(2n+1)^2} \quad \text{for} \quad \begin{cases} -a \leq x, \xi \leq a \\ \tau_0 \leq \tau \leq \infty \end{cases}$$

It is known that the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ is convergent, then inequality (2) implies that the series $\sum_{n=0}^{\infty} e^{-\mu_n^2\tau} \cos\mu_n x \cos\mu_n \xi$ should converge uniformly for $-a \leq x, \xi \leq a$ and $\tau_0 \leq \tau \leq \infty$.

Let $$S(x,\xi,\tau) = \sum_{n=0}^{\infty} e^{-\mu_n^2\tau}\cos\mu_n x \cos\mu_n \xi$$

Since each term of the series above is continuous, the function $S(x,\xi,\tau)$ is also continuous and bounded [11] for $-a \leq x, \xi \leq a$ and $\tau_o \leq \tau \leq \infty$.

APPENDIX B

1. Calculation of the Coefficient $A_5(x)$ in Equation (4.100)

Assume $K = \mu_o^2 = (\frac{\pi}{2a})^2$

From equation (4.101);

$$A_5(x) = \int_{-a}^{a} A_3(x,\xi)\,d\xi + \int_{-a}^{a} A_4(x,\xi)\,d\xi$$

where

$$A_3(x,\xi) = \frac{A_1(x,\xi)}{1+a_1(\xi)} = -\frac{2\alpha}{aK} B_o^2 \mu_o^2 \frac{\cos\mu_o x \cos\mu_o\xi \sin^2\mu_o\xi}{1+\frac{\alpha}{K} B_o \cos\mu_o\xi} \tag{1}$$

and

$$A_4(x,\xi) = \frac{A_2(1+a_1)}{\lambda_c} = \frac{\beta^*}{a\lambda_c} B_o \cos\mu_o x \cos^2\mu_o\xi \left[1+\frac{\alpha}{K} B_o \cos\mu_o\xi\right] \tag{2}$$

Let us evaluate integral $\int_{-a}^{a} A_3(x,\xi)d\xi$

$$\int_{-a}^{a} A_3(x,\xi)\,d\xi = -\frac{2\alpha}{aK} B_o^2 \mu_o^2 \cos\mu_o x \int_{-a}^{a} \frac{\cos\mu_o\xi \sin^2\mu_o\xi}{1+\frac{\alpha}{K} B_o \cos\mu_o\xi}\,d\xi$$

$$= -\frac{4\alpha}{aK} B_o^2 \mu_o^2 \cos\mu_o x \int_{0}^{a} \frac{\cos\mu_o\xi \sin^2\mu_o\xi}{1+\frac{\alpha}{K} B_o \cos\mu_o\xi}\,d\xi \tag{3}$$

Let
$$p = \frac{K}{\alpha B_o} \tag{4}$$

$$y = \cos\mu_o\xi$$

$$dy = -\mu_o \sin\mu_o\xi\,d\xi$$

Equation (3) becomes:

$$\int_{-a}^{a} A_3(x,\xi)\,d\xi = -\frac{4B_0}{a}\mu_0 \cos\mu_0 x \int_0^1 \frac{y\sqrt{1-y^2}}{y+p}\,dy \qquad (5)$$

The integral in the right hand side of equation (5) is separated into 2 different integrals as follows:

$$\int_0^1 \frac{y\sqrt{1-y^2}}{y+p}\,dy = \int_0^1 \frac{(y+p-p)\sqrt{1-y^2}}{y+p}\,dy$$

$$= \int_0^1 \sqrt{1-y^2}\,dy - p\int_0^1 \frac{\sqrt{1-y^2}}{y+p}\,dy \qquad (6)$$

But the first integral in the right hand side of equation (6) can be calculated from C.R.C. tables,

$$\int_0^1 \sqrt{1-y^2}\,dy = \frac{\pi}{4} \qquad (7)$$

The second integral can be evaluated after changing of variable:

Let $\qquad x = y + p$

$$dx = dy$$

hence,

$$\int_0^1 \frac{\sqrt{1-y^2}}{y+p}\,dy = \int_p^{p+1} \frac{\sqrt{-x^2+2px+1-p^2}}{x}\,dx \qquad (8)$$

Since it is obtained from any integral table that:

$$\int \frac{\sqrt{-x^2+2px+1-p^2}}{x}\,dx = \sqrt{-x^2+2px+1-p^2} + p\int \frac{dx}{\sqrt{-x^2+2px+1-p^2}}$$

$$+ (1-p^2)\int \frac{dx}{x\sqrt{-x^2+2px+1-p^2}} \qquad (9)$$

and $$\int \frac{dx}{\sqrt{-x^2+2px+1-p^2}} = -\sin^{-1}\left(\frac{-2x+2p}{\sqrt{4}}\right) = \sin^{-1}(x-p)$$

$$= \sin^{-1} y \qquad (10)$$

and $$\int \frac{dx}{x\sqrt{-x^2+2px+1-p^2}} = \frac{1}{\sqrt{p^2-1}}\sin^{-1}\left(\frac{px+1-p^2}{x}\right)$$

$$= \frac{1}{\sqrt{p^2-1}}\sin^{-1}\left(\frac{py+1}{y+p}\right) \qquad (11)$$

Substitute equation (10) and equation (11) into equation (9) to obtain:

$$\int \frac{\sqrt{-x^2+2px+1-p^2}}{x}\,dx = \sqrt{1-y^2} + p\sin^{-1}y - \sqrt{p^2-1}\,\sin^{-1}\left(\frac{py+1}{y+p}\right) \qquad (12)$$

which is integral (8). Thus we have

$$\int_0^1 \frac{\sqrt{1-y^2}}{y+p}\,dy = -1 + \frac{\pi}{2}p - \frac{\pi}{2}\sqrt{p^2-1} + \sqrt{p^2-1}\,\sin^{-1}\frac{1}{p} \qquad (13)$$

Since $p = \frac{K}{\alpha B_o}$ is very large, hence,

$$\frac{1}{p} \ll 1$$

and $$\sqrt{p^2-1} = p\left(1-\frac{1}{p^2}\right)^{\frac{1}{2}} \simeq p - \frac{1}{2p} - \frac{1}{8p^3}$$

$$\sin^{-1}\frac{1}{p} \simeq \frac{1}{p} + \frac{1}{6p^3}$$

Equation (13) can be written as follows:

$$\int_0^1 \frac{\sqrt{1-y^2}}{y+p}\,dy = -1 + \frac{\pi}{2}p - \frac{\pi}{2}\left(p - \frac{1}{2p} - \frac{1}{8p^2}\right) + \left(p - \frac{1}{2p} - \frac{1}{8p^3}\right)\left(\frac{1}{p} + \frac{1}{6p^3}\right)$$

$$\simeq \frac{\pi}{4p} - \frac{1}{3p^2} \qquad (14)$$

Now, inserting equation (7) and equation (14) into equation (6) yields:

$$\int_0^1 \frac{y\sqrt{1-y^2}}{y+p}\,dy = \frac{\pi}{4} - p\left(\frac{\pi}{4p} - \frac{1}{3p^2}\right) = \frac{1}{3p} = \frac{\alpha B_0}{3K}$$

which is the value of the integral in the right hand side of equation (5). Thus,

$$\int_{-a}^{a} A_3(x,\xi)\,d\xi = -\frac{4}{3a}\mu_0 \frac{\alpha}{K} B_0^2 \cos\mu_0 x$$

But $\mu_o = \frac{\pi}{2a}$

Then

$$\int_{-a}^{a} A_3(x,\xi)\,d\xi = -\frac{2}{3}\frac{\pi}{a^2}\frac{\alpha}{K} B_0^2 \cos\mu_0 x \qquad (15)$$

Next, evaluate integral $\int_{-a}^{a} A_4(x,\xi)d\xi$ where $A_4(x,\xi)$ is given by equation (2).

$$\int_{-a}^{a} A_4(x,\xi)\,d\xi = \frac{\beta^*}{a\lambda_c} B_0 \cos\mu_0 x \int_{-a}^{a}\left(1 + \frac{\alpha}{K}B_0\cos\mu_0\xi\right)\cos^2\mu_0\xi\,d\xi$$

$$= \frac{\beta^*}{a\lambda_c} B_0 \cos\mu_0 x\left[\int_{-a}^{a}\cos^2\mu_0\xi\,d\xi + \frac{\alpha}{K}B_0\int_{-a}^{a}\cos^3\mu_0\xi\,d\xi\right] \quad (16)$$

Let $y = \mu_o \xi = \frac{\pi}{2a} \xi$

$dy = \frac{\pi}{2a} d\xi$

hence,

$$\int_{-a}^{a} \cos^2 \mu_o \xi \, d\xi = 2\int_{-a}^{a} \cos^2 \mu_o \xi \, d\xi = \frac{4a}{\pi} \int_{0}^{\frac{\pi}{2}} \cos^2 y \, dy$$

$$= \frac{4a}{\pi} \left(\frac{\pi}{4} \right) = a \qquad (17)$$

and similarly,

$$\int_{-a}^{a} \cos^3 \mu_o \xi \, d\xi = \frac{4a}{\pi} \int_{0}^{\frac{\pi}{2}} \cos^3 y \, dy = \frac{4a}{\pi} \left(\frac{2}{3} \right)$$

$$= \frac{8}{3} \frac{a}{\pi} \qquad (18)$$

Substituting equations (17) and (18) into equation (16) gives

$$\int_{-a}^{a} A_4(x, \xi) \, d\xi = \frac{\beta^*}{a\lambda_c} B_o \cos\mu_o x \left[a + \frac{8}{3} \frac{\alpha}{K} \frac{a}{\pi} B_o \right]$$

$$= \left[\frac{\beta^*}{\lambda_c} + \frac{8}{3} \frac{\beta^* \alpha B_o}{\pi K \lambda_c} \right] B_o \cos\mu_o x \qquad (19)$$

Thus, $A_5(x)$ is obtained by summing up equations (15) and (19)

$$A_5(x) = -\frac{2}{3} \frac{\pi}{a^2} \frac{\alpha}{K} B_o^2 \cos\mu_o x + \left(\frac{\beta^*}{\lambda_c} + \frac{8}{3} \frac{\beta^* \alpha B_o}{\pi K \lambda_c} \right) B_o \cos\mu_o x$$

or

$$A_5(x) = \left(\frac{\beta^*}{\lambda_c} + \frac{8}{3}\frac{\beta^*\alpha B_0}{\pi K \lambda_c} - \frac{2}{3}\frac{\pi}{a^2}\frac{\alpha}{K}B_0\right)B_0 \cos\mu_0 x \quad (20)$$

Let us consider the order of magnitude of each term in brackets. To do this, we assume the following data [13]

$a = 5$ $\qquad$ $\beta = 0.0033$

$\Sigma_f = 0.00267 \text{cm}^{-1}$ $\qquad$ $\Sigma_a = 0.0052 \text{ cm}^{-1}$

$\rho = 0.0697 \text{ g/cm}^3$ $\qquad$ $c_p = 0.078 \text{ cal/g}^\circ\text{K}$

$v = 4.861 \times 10^7 \text{cm/sec}$ $\qquad$ $\alpha_0 = -5\times10^{-5} \text{ }^\circ\text{K}^{-1}$

$\lambda = 0.824 \text{ sec}^{-1}$ $\qquad$ $e = 200 \frac{\text{Mev}}{\text{fission}} = 200\times3.82\times10^{-14}\frac{\text{cal}}{\text{fission}}$

Therefore, we obtain:

$$K = \left(\frac{\pi}{2a}\right)^2 = \left(\frac{\pi}{10}\right)^2 = 0.0986$$

$$K \simeq k_0 - 1 \implies k_0 \simeq 1.0986$$

$$\gamma_2 = \frac{e\Sigma_f}{v\Sigma_a \rho C_p} = 1.1\times10^{-16} \text{ cm}^2 \text{sec } ^\circ\text{K}$$

$$\alpha = |(1-\beta)\alpha_0\gamma_2| \simeq |\alpha_0\gamma_2| = 5.5\times10^{-21} \text{ cm}^2 \text{sec}$$

$$\lambda_c = \frac{\lambda}{v\Sigma_a} = \frac{0.824}{4.861\times10^7\times0.0052} = 3.2\times10^{-6} \text{ (dimensionless)}$$

$$\frac{8}{3}\frac{\alpha}{\pi K} = 4.7\times10^{-20} \text{ cm}^2 \text{sec}.$$

$$\frac{\beta^*}{\lambda_c} = \beta k \simeq \beta k_o = 0.00362 \qquad (21)$$

$$\frac{8}{3}\frac{\beta^*\alpha}{\pi K\lambda_c} = \left(\frac{8}{3}\frac{\alpha}{\pi K}\right)\left(\frac{\beta^*}{\lambda_c}\right) = 1.7\times10^{-22}\ cm^2 sec \qquad (22)$$

$$\frac{2}{3}\frac{\pi}{a^2}\frac{\alpha}{K} = 4.6\times10^{-21}\ cm^2\ sec \qquad (23)$$

$$\beta^* = 0.00362\lambda_c = 0.00362\times3.2\times10^{-6} = 1.1\times10^{-8}$$

Thus, comparing equation (21) to equations (22) and (23) yields:

$$-\frac{8}{3}\frac{\beta^*\alpha B_o}{\pi K\lambda_c} \ll \frac{\beta^*}{\lambda_c}$$

and

$$\frac{2}{3}\frac{\pi}{a^2}\frac{\alpha}{K} \ll \frac{\beta^*}{\lambda_c}$$

and the value of $A_5(x)$ is approximately equal to

$$A_5(x) \simeq \frac{\beta^*}{\lambda_c} B_o \cos\frac{\pi}{2a}x \qquad (24)$$

Hence, $A_5(x)$ is finite and positive for $-a < x < a$.

2. Calculation of the Coefficient $A_8(x)$ in Equation (4.114)

Assume $K > \mu_o^2$

From equation (4.115):

$$A_8(x) = \int_{-a}^{a} A_7(x,\xi)\, d\xi$$

where $A_7 = \frac{A_2 a_1}{\lambda_c (K-\mu_o^2)} = \frac{\beta^* \alpha B_o^2}{aK\lambda_c (K-\mu_o^2)} \cos\mu_o x \cos^3(\mu_o \xi)$

$$A_8(x) = \frac{\beta^* \alpha B_o^2}{aK\lambda_c (K-\mu_o^2)} \cos\mu_o x \int_{-a}^{a} \cos^3(\mu_o \xi)\, d\xi \qquad (25)$$

Since the value of the integral in the right hand side of expression (25) is given by equation (18), hence,

$$A_8(x) = \frac{8}{3} \frac{\beta^* \alpha B_o^2}{\pi K\lambda_c (K-\mu_o^2)} \cos \frac{\pi}{2a} x \qquad (26)$$

The equation above shows that $A_8(x)$ is finite and positive for $-a < x < a$.

BIBLIOGRAPHY

1. W. E. Kastenberg and P. L. Chambré, "On the Stability of Nonlinear Space-dependent Reactor Kinetics," Nucl. Sci. Eng. 31, 67-79 (1968).

2. H. L. Garabedian and R. E. Lynch, "Nonlinear Reactor Kinetic Analysis," Nucl. Sci. Eng. 21, 550 (1965).

3. R. Scalettar, "Space and Energy Dependent Corrections to the Fuchs-Nordheim Model," GA 3400, General Atomic (August 1962).

4. S. Kaplan, S. G. Margolis and D. R. Harris, "Transient Response of the Special Flux Distribution in an Initially Uniform Bare Reactor to Arbitrary Nonuniform Changes in Materials Properties," Westinghouse Bettis Technical Report, WAPD-BT-1, pp. 69-83, (1960).

5. S. Kaplan, A. F. Hency, S. G. Margolis and J. J. Taylor, "Space-time Reactor Dynamics," Proc. Intern. Conf. Peaceful Uses Atomic Energy, Geneva, 4, pp. 41-53, Intern. Atomic Energy Agency (1964).

6. R. V. Meghreblian and D. K. Holmes, "Reactor Analysis," 1st edi tion, McGraw Hill (1960).

7. D. L. Hetrik, "Dynamics of Nuclear Reactors," University of Chicago Press (1971).

8. H. E. Wilhelm, "Nonlinear Initial-Boundary-Value Problem for Convection, Diffusion, Ionization and Recombination Processes," J. Math. Phys. 13, 252-256, February (1972).

9. A. N. Tikhonov and A. A. Samarskii, "Equations of Math. Physics," pp. 221-300 (1963).

10. J. W. Dettman, "Maths. Methods in Phys. and Engineering," 1st edition, pp. 190-217 (1962).

11. E. Kreyszig, "Advanced Eng. Math.," pp. 546-587, 3rd edition, (1972).

12. R. V. Churchill, "Fourier Series and Boundary Value Problems," pp. 186-187, 2nd edition (1963).

13. K. A. Solomon and W. E. Kastenberg, "Linear Stability of Fast and Thermal Reactor Models Using Space-Time Kinetics," Nucl. Sci. and Eng. 49, 99-108 (1972).

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
2.	Chairman, Code 59 Department of Mechanical Engineering Naval Postgraduate School Monterey, California 93940	1
3.	Professor D. H. Nguyen, Code 59Ng Department of Mechanical Engineering Naval Postgraduate School Monterey, California 93940	2
4.	LT. Ng. v. Minh 229 Doc-Lap, Nha-Trang South Vietnam	3
5.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

REPORT TITLE

SOLUTION FOR A NONLINEAR NUCLEAR REACTOR with Negative prompt feedback and One-group delayed neutron

DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Master's Thesis; (June 1973)

AUTHOR(S) (First name, middle initial, last name)

Nguyen van Minh; Lieutenant, Vietnamese Navy

REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1973	89	13

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

The nonlinear space-time neutron flux equation with negative prompt feedback and one-group delayed neutron is reduced by the use of a nonlinear transformation to a partial differential equation, in which the nonlinear term represents a small perturbation. The general procedure of solution for the resulting weakly non-linear initial-boundary-value problem is then established by means of the method of successive approximation. Convergence of the analytical solution is also discussed. The solutions to a slab reactor core and a cylindrical reactor core are investigated here. Asymptotic stable equilibrium states are derived from each of these solutions. The present results are consistent with those obtained from previous stability analysis for the generalized buckling K greater or less than μ_0^2, the lowest eigen value of the associated linear HELMHOLTZ equation.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified
Security Classification

A-31408

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

Thesis 145183
M5985 Minh
c.1 Solution for a non-linear nuclear reactor with negative prompt feedback and one-group neutron.